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Completeness of the Bethe Ansatz for the Six and Eight-Vertex Models

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We discuss some of the difficulties that have been mentioned in the literature in connection with the Bethe ansatz for the six-vertex model and XXZ chain, and for the eight-vertex model. In particular we discuss the "beyond the equator," infinite momenta and exact complete string problems. We show how they can be overcome and conclude that the coordinate Bethe ansatz does indeed give a complete set of states, as expected.

KEY WORDS: Statistical mechanics; six-vertex model; eight-vertex model; Bethe Ansatz; completeness.

1. INTRODUCTION

There are proofs in the literature of the combinatorial completeness of the Bethe ansatz[1] for the six-vertex model and XXZ chain:[2, 3] i.e., that for a lattice of N columns it gives all 2^N eigenvectors (states) of the transfer matrix. In the presence of an arbitrary field H there seems to be no doubt that this is so, and this is in agreement with the results of the numerical experiments we report in Section 2. Other studies have been made, also indicating the completeness of the Bethe ansatz.[4–8]

Even so, there still appear papers that either question this completeness, or at least appear to question it, when the field is zero, or at special values of the crossing parameter λ (or η). Statements have been made that "the Bethe vector vanishes" for states with more down arrows than up arrows,[9, 10] and that it is incomplete or "singular" if some of the momenta are infinite.[11–13] Here we show that these problems can be overcome in the

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coordinate Bethe ansatz if one recognizes that one is dealing with a set of algebraic equations, so must include an appropriately generalized "point at infinity" in one's considerations, and properly normalize the eigenvector.

Recently it has been claimed that "Bethe's equation is incomplete" at special "roots of unity" values of λ.[14–17] By this it is meant that the Bethe zeros $v_1,\ldots, v_n$, or equivalently the Bethe momenta $k_1,\ldots, k_n$, are not uniqely defined. They contain at least one exact complete string, and one is free to choose each string centre at will.[2] This freedom is noted in ref. 17, after Eq. (1.35) therein. We show that this freedom is because the eigenvalue is degenerate, so the eigenvector itself is not uniqely defined. Any allowed choice of $v_1,\ldots, v_n$ gives a valid eigenvector. The set of such choices is a curve in the eigenspace. For the simplest case, which is when $v_1,\ldots, v_n$ form just one single string, we show that the vectors on this curve span the eigenspace. Thus the Bethe ansatz is complete for this case, precisely because of this lack of uniqueness. We fully expect this argument to generalize to more complicated cases.

In all the cases we have looked at, we have found that the Bethe ansatz is indeed complete: it gives all the eigenvectors (states). More precisely, it can be used to construct a basis for each eigenspace.[3]

Apart from the difficulty mentioned in Section 6, the problems we encounter can be resolved by the methods mentioned after Eq. (46) and reviewed in the summary.

We also present the coordinate Bethe ansatz equations for the eight vertex model in zero field, with an even number of columns, and discuss how the infinite momenta and exact complete string problems can be resolved. We expect these equations to be similarly complete.

We further show that the functional relation between the eigenvalues $T(v)$, $Q(v)$ of the T and Q matrices can itself be written as a generalized eigenvalue problem, with the Fourier coefficients of $Q(v)$ being the elements of the eigenvector.

In one sense these problems have to be resolvable, since if the ansatz is complete for arbitrary field H and crossing parameter λ, then (at least in principle) one can always deal with difficult cases by taking a limit. (When λ is real or pure imaginary, and the field H is zero or pure imaginary, then one can choose the spectral variable v so that the transfer matrix is hermitian, so we know it is diagonalizable. Since the eigenvectors are

[2] Except that if $H=0$, N is even and $n=N/2$, then one can require that the eigenvector g also be an eigenvector of the arrow reversal operator R and that $v_1,\ldots, v_n$ satisfy the constraint (151).

[3] This contradicts the statement in the abstract of ref. 15, and repeated in the introduction of ref. 18, that "the Bethe ansatz equations determine only the eigenvectors which are the highest weights of the infinite dimensional sl_2 loop algebra."

independent of v, this means that it is diagonalizable for all v.) However, the question is whether one can do better than this and find all the eigenvectors for any particular values of H and λ. For the problems that have been addressed in the literature, we believe the answer to be yes.

2. THE SIX-VERTEX MODEL IN A FIELD

The six-vertex model[(19–27)] has Boltzmann weights $\omega_1,\dots, \omega_6$. Let

$$R_{++} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_4 \end{pmatrix}, \qquad R_{+-} = \begin{pmatrix} 0 & 0 \\ \omega_6 & 0 \end{pmatrix}$$
$$R_{-+} = \begin{pmatrix} 0 & \omega_5 \\ 0 & 0 \end{pmatrix}, \qquad R_{--} = \begin{pmatrix} \omega_3 & 0 \\ 0 & \omega_2 \end{pmatrix} \tag{1}$$

Let $\sigma = \sigma_1,\dots, \sigma_N$ denote the state of a row of N vertical arrows ($+1$ for an up arrow, -1 for a down arrow). Then the transfer matrix for a lattice of N rows is the 2^N by 2^N matrix T with elements

$$T_{\sigma,\sigma'} = \text{Trace } R_{\sigma_1,\sigma_1'} R_{\sigma_2,\sigma_2'} \cdots R_{\sigma_N,\sigma_N'} \tag{2}$$

Considered as a function of $\omega_1,\dots, \omega_6$, it has the symmetries:

$$T(\omega_1,\dots, \omega_6) = (-1)^N T(-\omega_1,\dots, -\omega_6) = T^t(\omega_4, \omega_3, \omega_2, \omega_1, \omega_6, \omega_5) \tag{3}$$

the superfix t denoting transposition.

In statistical mechanics one wants to calculate the partition function

$$Z = \text{Trace } T^{M_r} \tag{4}$$

where M_r is the number of rows of the lattice. It is therefore desirable to diagonalize the matrix T. This problem has been solved (for general values of $\omega_1,\dots, \omega_6$) by the Bethe ansatz.[(23–26)] [4]

In the Bethe ansatz one characterizes the state $\sigma_1,\dots, \sigma_N$ by the positions $X = x_1,\dots, x_n$ of the down arrows, i.e., $\sigma_j = -1$ iff one of $x_1,\dots, x_n$ is equal to j, else $\sigma_j = +1$. Because of the "ice rule" that there be two arrows into each vertex, and two arrows out, the number n is conserved, being the same on all rows of the lattice. For a lattice of N columns, the transfer

[4] In fact, ref. 26 is more general yet, considering an inhomogeneous model where the field H and the rapidity variable v vary from column to column. Here we shall only consider the homogeneous model.

matrix T therefore breaks up into $N+1$ diagonal blocks, one for each value of n from 0 to N. Within block n, an eigenvector g has elements $g(X) = g(x_1,\ldots, x_n)$ for each state σ or X.

The Bethe ansatz is the following guess at the elements of the eigenvector:

$$g(x_1,\ldots, x_n) = \sum_P A(p_1,\ldots, p_n)\, e^{ik_{p_1}x_1} \cdots e^{ik_{p_n}x_n} \tag{5}$$

where the sum is over all the $n!$ permutations $\{p_1,\ldots, p_n\}$ of $\{1,\ldots, n\}$, and the integers $x_1,\ldots, x_n$ lie in the range

$$1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant N \tag{6}$$

Substituting this ansatz directly into the eigenvalue/eigenvector equations, one finds the following sufficient conditions for g to be an eigenvector:

$$s_{p_j,\, p_{j+1}} A(p_1,\ldots, p_n) + s_{p_{j+1},\, p_j} A(p_1,\ldots, p_{j+1}, p_j,\ldots, p_n) = 0 \tag{7}$$

$$e^{iNk_{p_1}} A(p_2,\ldots, p_n, p_1) = A(p_1,\ldots, p_n) \tag{8}$$

for $j = 1,\ldots, n-1$ and all permutations $\{p_1,\ldots, p_n\}$.

Here

$$s_{j,\, m} = \omega_1\omega_3 - (\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6)\, e^{ik_m} + \omega_2\omega_4 e^{i(k_j + k_m)} \tag{9}$$

We emphasize that (5)–(9) are *sufficient* conditions (together with $g \neq 0$) for g to be an eigenvector. They involve the Boltzmann weights only via the ratios

$$(\omega_1\omega_2 + \omega_3\omega_4 - \omega_5\omega_6)/(\omega_1\omega_3) \qquad \text{and} \qquad \omega_2\omega_4/(\omega_1\omega_3) \tag{10}$$

This implies that the transfer matrices of two models with different weights, but the same values of these ratios, *commute*. This can be proved directly by appropriately extending the method of Section 9.6 of ref. 27, or of refs. 28 or 29).

The eigenvalue corresponding to this eigenvector is

$$\Lambda = \omega_1^N \prod_{j=1}^{n} \frac{\omega_1\omega_3 + (\omega_5\omega_6 - \omega_3\omega_4)\, e^{ik_j}}{\omega_1(\omega_1 - \omega_4 e^{ik_j})} + \omega_4^N \prod_{j=1}^{n} \frac{\omega_1\omega_2 - \omega_5\omega_6 - \omega_2\omega_4 e^{ik_j}}{\omega_4(\omega_1 - \omega_4 e^{ik_j})} \tag{11}$$

If the weights $\omega_1,\ldots,\omega_6$ are all non-zero, it is convenient to write them as

$$\omega_1 = e^{H+V}a, \qquad \omega_2 = e^{-H-V}a, \qquad \omega_3 = e^{H-V}b$$
$$\omega_4 = e^{-H+V}b, \qquad \omega_5 = \omega_6 = c \tag{12}$$

Here H is a dimensionless electric field in the horizontal direction; V is the field in the vertical direction.[5]

Defining

$$\Delta = (a^2+b^2-c^2)/(2ab), \qquad e^{i\kappa_j} = e^{-2H}e^{ik_j} \tag{13}$$

and dividing s_{jm} by a constant factor $e^{2H}ab$ that cancels out of (7), the above equations involving $\omega_1,\ldots,\omega_6$ simplify to

$$s_{j,m} = 1 - 2\Delta e^{i\kappa_m} + e^{i(\kappa_j+\kappa_m)} \tag{14}$$

$$\Lambda = e^{(N-2n)V}\left\{ e^{NH}\prod_{j=1}^{n}\frac{ab+(c^2-b^2)\,e^{i\kappa_j}}{a(a-be^{i\kappa_j})} + e^{-NH}\prod_{j=1}^{n}\frac{a^2-c^2-abe^{i\kappa_j}}{b(a-be^{i\kappa_j})}\right\} \tag{15}$$

Note that V enters the above equations only via a factor $e^{(N-2n)V}$ in the eigenvalue Λ. This is because of the ice rule: T commutes with the diagonal matrix S_z which has entries $(\sigma_1+\sigma_2+\cdots+\sigma_N)\,\delta_{\sigma,\sigma'}$. The effect of V in the Boltzmann weights (2) is to pre- and post-multiply T by $\exp(S_zV/2)$. Within the diagonal block n, the diagonal entries of S_z are $N-2n$, so the effect of introducing V is to merely multiply all entries of T by $e^{(N-2n)V}$. Without loss of generality we shall from now on take $V=0$.

The next step is to derive what Fabricius and McCoy call "Bethe's equation" [15, Eq. (1.2)]. This is easily done when the $s_{j,m}$ are all non-zero, but much of the apparent confusion in the literature arises when some of them are zero, so we proceed carefully. To be an eigenvector, g must be non-zero, so at least one of the coefficients $A(p_1,\ldots,p_n)$ must be non-zero. In Appendix A we show that this implies that

$$e^{iNk_j}\prod_{m=1,\,m\neq j}^{n} s_{j,m} = (-1)^{n-1}\prod_{m=1,\,m\neq j}^{n} s_{m,j} \tag{16}$$

for $j=1,\ldots,n$. Hence these n equations are a necessary consequence of the linear equations (7) and (8) for the $n!$ coefficients $A(p_1,\ldots,p_n)$. We discuss below their sufficiency for the case when all of the $s_{j,m}$ are non-zero.

[5] The weights ω_5, ω_6 are sinks and sources of horizontal arrows, so only occur in the partition function and transfer matrix in the product combination $\omega_5\omega_6$. This means that there is no loss of generality in choosing $\omega_5=\omega_6=c$.

Also, substituting all n cyclic permutations of $\{p_1,\ldots, p_n\}$ into (8), one gets n linear homogeneous equations for the n coefficients A that occur. Since at least one of them is non-zero (for some permutation $\{p_1,\ldots, p_n\}$), the determinant of coefficients must vanish, giving

$$e^{iN(k_1+k_2+\cdots+k_n)} = 1 \tag{17}$$

We refer to (5)–(17) as "the Bethe ansatz equations."

Transformation to Difference Variables

Define ρ, λ, v so that

$$\begin{aligned} a &= \rho \sinh[(\lambda - v)/2] \\ b &= \rho \sinh[(\lambda + v)/2] \\ c &= \rho \sinh \lambda \end{aligned} \tag{18}$$

then

$$\Delta = -\cosh \lambda \tag{19}$$

Define also $v_1,\ldots, v_n$ so that

$$e^{ik_j} = e^{2H} e^{i\kappa_j} = e^{2H} \frac{e^{\lambda} - e^{v_j}}{e^{\lambda + v_j} - 1} \tag{20}$$

Then

$$s_{i,j} = \frac{\sinh \lambda \sinh[(v_i - v_j + 2\lambda)/2]}{\sinh[(v_i + \lambda)/2] \sinh[(v_j + \lambda)/2]} \tag{21}$$

If we define functions $\phi(v)$, $Q(v)$ by

$$\begin{aligned} \phi(v) &= \rho^N \sinh^N(v/2) \\ Q(v) &= \prod_{j=1}^{n} \sinh[(v - v_j)/2] \end{aligned} \tag{22}$$

then (15) can be written in the form

$$\Lambda = (-1)^n \frac{e^{NH} \phi(\lambda - v)\, Q(v + 2\lambda) + e^{-NH} \phi(\lambda + v)\, Q(v - 2\lambda)}{Q(v)} \tag{23}$$

We shall also introduce a parameter q by

$$q = -e^{\lambda}, \qquad \Delta = (q + q^{-1})/2 \tag{24}$$

This q is the $-q$ of ref. 15.

Commuting Transfer Matrices

In the notation (18)–(19), the ratios (10) that enter the eigenvector calculation are

$$2e^{-2H}\Delta \qquad \text{and} \qquad e^{-4H}$$

Keeping ρ, λ, H fixed, and writing the transfer matrix T as a function $T(v)$ of v, it follows that

$$T(v)\, T(v') = T(v')\, T(v) \tag{25}$$

for all v, v'. The transfer matrices $T(v)$, $T(v')$ *commute.*

Further, if we define the Pauli matrices

$$\sigma_j^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_j^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_j^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{26}$$

acting on the spin (arrow) in position j, then the logaritmic derivative of $T(v)$ at $v = -\lambda$ is a linear combination of the identity operator and the hamiltonian

$$\begin{aligned} \mathscr{H} = -\tfrac{1}{2} \sum_{j=1}^{N} \{ &\cosh H(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \\ &- i \sinh H(\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x) + \Delta \sigma_j^z \sigma_{j+1}^z \} \end{aligned}$$

interpreting suffixes $N+1$ as 1. Hence $T(v)$ also commutes with this hamiltonian.

Incrementing v by $2\pi i$ is the same as negating all of a, b, c.[6] This merely multiplies T by $(-1)^N$. The lowest and highest powers of $e^{v/2}$ that can occur in the expansion of an element of $T(v)$ are $-N$ and N. It follows that $T(v)$ can be expanded in the form

$$T(v) = \sum_{r=0}^{N} T_r e^{(N-2r)\, v/2} \tag{27}$$

the coefficients T_r being matrices independent of v.

[6] Remember that ω_5 and ω_6 always occur in pairs, so negating c has no effect on T.

The commutation relations (25) imply that $T_0,\ldots, T_N$ all commute with one another, and with $\mathscr{H}$.

Hermiticity of T and Unitarity of P

Negating v and H interchanges ω_1 with ω_4, and ω_2 with ω_3. From (3), this transposes the transfer matrix T. If λ, ρ are real and H, v are pure imaginary, or if v is real and ρ, H, λ are pure imaginary, this implies that T is hermitian. Thus it is diagonalizable. Combining this with the commutation properties above, it follows that there exists a unitary eigenvector matrix P such that

$$P^{\dagger}T_r P = \text{diagonal} \qquad \text{and} \qquad P^{\dagger}\mathscr{H}P = \text{diagonal} \tag{28}$$

In both cases Δ is real. For these cases, the eigenvalue Λ must be real. Since H, v/λ are pure imaginary in both, (23) suggests that $Q(v+2\lambda)^* = Q(v-2\lambda)$. From (22), this implies that:

(i) λ real: $v_1,\ldots, v_n$ are either pure imaginary or occur in pairs v_j, $-v_j^*$ positioned symmetrically about the imaginary axis;

(ii) λ pure imaginary: $v_1,\ldots, v_n$ are either real or occur in complex conjugate pair.

In both cases it follows that the wave numbers $k_1,\ldots, k_n$ are either real or occur in complex conjugate pairs. Similarly for $\kappa_1,\ldots, \kappa_n$.

Note that P is independent of v, so provided H is pure imaginary, we have proved that $T = T(v)$ is diagonalizable and P is unitary for *all* complex v.

Of course, we usually take the Boltzmann weights $\omega_1,\ldots, \omega_6$ to be positive real, so from that point of view we would like to take H to be real, rather than pure imaginary. However, it is reassuring to have a proof that T is diagonalizable even if the proof only holds when H is pure imaginary. This does include the "zero-field" case of main interest, as well as other problems thta have been looked at, such as the critical Potts model with $q < 4$.[7]

This implies that T will be diagonalizable for all H except posssibly for isolated non-zero values off the imaginary axis. Our question is: does the Bethe ansatz give all the eigenvectors?

[7] In Section 6.2 of ref. 30 it is shown that this model is equivalent to a six-vertex model with a boundary seam. This seam is equivalent to introducing a horizontal field $H = \theta/N$, where $q^{1/2} = 2\cosh\theta$, so θ and H are pure imaginary when $q < 4$ and the model is critical.

The matrix $\tilde{Q}(v)$

Each eigenvalue Λ must have an expansion corresponding to (27) :

$$\Lambda(v) = \sum_{r=0}^{N} t_r e^{(N-2r)\, v/2} \tag{29}$$

Thus Λ is an entire function of v and the RHS of (23) must vanish when the denominator does, giving

$$e^{NH}\phi(\lambda - v_j)\, Q(v_j + 2\lambda) + e^{-NH}\phi(\lambda + v_j)\, Q(v_j - 2\lambda) = 0 \tag{30}$$

for $j = 1,..., n$.

These are precisely Eqs. (16). The author noted (for $H = 0$) in 1971 that they imply the existence of a matrix $\tilde{Q}(v)$ with eigenvalues $Q(v)$ that commutes with $T(v)$ and satisfies the matrix functional relation

$$T(v)\, \tilde{Q}(v) = e^{NH}\phi(\lambda - v)\, \tilde{Q}(v + 2\lambda') + e^{-NH}\phi(\lambda + v)\, \tilde{Q}(v - 2\lambda') \tag{31}$$

where $\lambda' = \lambda - i\pi$. This proved to be the key to solving the eight-vertex model.[28, 29]

For the zero-field eight-vertex model an explicit construction for $\tilde{Q}(v)$ is given in Section 6 of ref. 31 and in Section (10.5) of ref. 27.[8] This is specialized to the six-vertex model in Eqs. (8), (96), and (97) of ref. 31, for $H = 0$ and N even.

The vectors $\Phi_n(v \mid \sigma)$ therein are generalizations of the special eigenvectors ψ we discuss in Section 3. They form the columns of a matrix $\Phi_n(v)$. Writing $[N, m] = N!/(m!(N-m)!)$, this matrix has $[N, n]$ rows and $[N, N/2]$ columns. Provided the rows are linearly independent, we can define the $[N, n]$ by $[N, n]$ matrix $\tilde{Q}(v)$ by

$$\tilde{Q}(v)\, \Phi_n(v_0 \mid \sigma) = \Phi_n(v \mid \sigma) \tag{32}$$

v_0 being an arbirary fixed parameter, for all n from 0 to N. Then it is shown in refs. 31 and 27 that

$$\tilde{Q}(v)\, T(v) = T(v)\, \tilde{Q}(v) \tag{33}$$

so we can simultaneously diagonalize both $T(v)$ and $\tilde{Q}(v)$. Doing this, we find (for $H = 0$ and N even) that the eigenvalues $Q(v)$ must indeed have the form given in Eq. (40) below.

[8] As we remark in Section 7, there is a typing error in Eq. (10.5.8) of ref. 27: σ_{j+1} therein should be σ_{j-1}.

In the cases discussed in Section 5, where some of $v_1,\ldots, v_n$ form one or more complete strings, the eigenvalue Λ of $T(v)$ is degenerate. This is reflected in the fact that the string centres (the average value of the v_j within a string) are not determined by the Bethe ansatz. One of the main objects of this paper is to stress that this is not a deficiency or "incompleteness" of the Bethe ansatz, but rather a strength. It means that it can be used to construct a complete basis of the eigenspace.

However, the corresponding eigenvalues of $\tilde{Q}(v)$ are *not* degenerate, so one can fix the string centres by requiring that g in (5) be also an eigenvector of $\tilde{Q}(v)$. This is what Fabricius and McCoy have achieved. In physicist's terms, they have resolved the degeneracy of the eigenvalue Λ; in mathematician's terms, they have made a particular choice of the basis of the eigenspace of $T(v)$.

Beyond the Equator: The Relation Between the n and N−n Solutions

As we note below, there appears to be no problem solving these equations in the presence of a non-zero field H (pure imaginary or real), even when $n > N/2$ and there are more down arrows than up. However, the eignvalues Λ are then the same as the mirror case (where all arrows are reversed), with $n \to N-n$ and $H \to -H$. Is there a relation between the two solutions?

This problem has been studied by Bazhanov *et al.*[32, 33] Consider Eq. (23) for some n and H, with a function $Q_1(v) = Q(v)$, and the same equation with n, H replaced by $N-n$, $-H$, with a different function $Q_2(v) = Q(v)$ but the same Λ. Eliminate Λ between the two equations. We obtain

$$W(v+\lambda) = W(v-\lambda) \tag{34}$$

where the "Wronskian" $W(v)$ is defined by

$$W(v) = \frac{e^{NH} Q_1(v+\lambda)\, Q_2(v-\lambda) - (-1)^N e^{-NH} Q_1(v-\lambda)\, Q_2(v+\lambda)}{\phi(v)} \tag{35}$$

We continue to require that $Q_1(v)$ and $Q_2(v)$ be of the form (22), with n replaced by n, $N-n$, respectively. It follows that

$$W(v+2\pi i) = W(v) \tag{36}$$

For arbitrary λ, excluding the "root of unity" cases discussed in Section 5 in which $i\lambda$ is a rational fraction of π, the only solution of both (34) and (36) is that $W(v)$ be a constant D. Hence for non-"root of unity" cases, $Q_1(v)$ (with parameters n, H), and $Q_2(v)$ (with parameters $N-n$, $-H$) must satisfy the relation

$$e^{NH}Q_1(v+\lambda)\,Q_2(v-\lambda)-(-1)^N\,e^{-NH}Q_1(v-\lambda)\,Q_2(v+\lambda)=D\phi(v) \tag{37}$$

For $H=0$, this is Eq. (47) of ref. 10. We shall discuss it further below, particularly for the case when $H=0$ and N is even.

Pronko and Stroganov[10] have also addressed this problem, in a different manner. Using their terminology, we shall show in Section 4 that it is indeed interesting, and perfectly possible, to consider Bethe's equations "on the wrong side of the equator."

Bethe's Equation as a Generalized Eigenvalue Problem

We can write (23) as

$$\Lambda(v)\,Q(v)=(-1)^n\,\{e^{NH}\phi(\lambda-v)\,Q(v+2\lambda)+e^{-NH}\phi(\lambda+v)\,Q(v-2\lambda)\} \tag{38}$$

and $\phi(v)$, $Q(v)$ as

$$\phi(v)=\sum_{r=0}^{N}f_r e^{(N-2r)\,v/2} \tag{39}$$

$$Q(v)=\sum_{j=0}^{n}q_j e^{(n-2j)\,v/2} \tag{40}$$

Substituting these expansions, together with (29), into the Λ, Q equation above, one obtains

$$\sum_{j=0}^{n}b_{i,j}q_j=0 \tag{41}$$

where if $0\leqslant i-j\leqslant N$,

$$b_{i,j}=-t_{i-j}+(-1)^n\,e^{\lambda N/2}[(-1)^N\,e^{NH}e^{\lambda(i-3j+n-N)}+e^{-NH}e^{\lambda(3j-i-n)}]\,f_{i-j} \tag{42}$$

else $b_{i,j}=0$. Here $i=0,\ldots,N+n$.

We see that we have $N+n+1$ equations for the $N+n+1$ unknowns $t_0,\ldots,t_N$ and $q_0:q_1:\cdots:q_n$. These are an alternative form of Bethe's equations. They define the eigenvalue $\Lambda(v)$ and (usually) the function $Q(v)$. They are linear in the t_j, and homogeneous and linear in the q_j. The t_j play the role of a set of "eigenvalues," q_j that of an "eigenvector."

This form of Bethe's equation has some advantages which we shall mention as we come to them in the following four sections. In particular, suppose that we actually know, or have guessed, the eigenvalue $\Lambda(v)$, and hence $t_0,\ldots, t_N$. Then the equations are a set of homogeneous linear equations for $q_0,\ldots, q_N$, and can be solved by the standard apparatus of linear algebra. Let **B** be the $N+n+1$ by $n+1$ matrix with elements b_{ij}. Then we can distinguish three cases:

(1) **B** has rank $n+1$: then there are no solutions for $q_0,\ldots, q_n$. $\Lambda(v)$ is not an eigenvalue.

(2) **B** has rank n: there is one solution for the ratios $q_0 : q_1 : \cdots : q_n$. This presumably means that the eigenvector g is unique: then $\Lambda(v)$ is an eigenvalue with degeneracy one.

(3) **B** has rank less than n: there is more than one solution for $q_0 : q_1 : \cdots : q_n$. The eigenvector g is presumably not unique: then $\Lambda(v)$ is an eigenvalue with degeneracy greater than one.

Thus we can use these simple considerations to determine whether a given eigenvalue is single or multiple.

"Bethe's Equations"

There are a huge number of equations in (7): $(n-1)\times n!/2$ homogeneous linear equations for the $n!$ coefficients $A(p_1,\ldots, p_n)$. Fortunately it seems that they always permit at least one (possible more—this is the source of some of the misunderstandings in the literature) non-identically zero solution. This is of the form

$$A(p_1,\ldots, p_n) = \epsilon_P C^{-1} \prod_{1\leqslant i<j\leqslant n} t_{p_j, p_i} \tag{43}$$

where $\epsilon_P = \pm 1$ is the sign of the permutation and the t_{ij} must satisfy

$$t_{ij} s_{ji} = t_{ji} s_{ij} \tag{44}$$

At least one of t_{ij} and t_{ji} must be non-zero, else the $A(p_1,\ldots, p_n)$ would all vanish and g would be the zero vector.

If the s_{ij} are all finite and non-zero and the k_j are finite, then we can take $t_{ij} = s_{ij}$ and choose the normalization factor C to be unity. The problems discussed in this paper arise when this is not so. (Apart from the equal v_j difficulty touched on in Section 6.) For these cases one should choose C so that the maximum term in the summand in (5) is finite and non-zero (say unity). This maximum is to be taken over all permutations P and all values of $x_1,..., x_n$ allowed by (6).

This is the solution given in (8.4.10) of ref. 27, except that there we took $t_{ij} = s_{ij}$ for all i, j. We wish to be more general here so as to cope with the situation when some of the s_{ij} vanish.

Substituting (43) into (8), we obtain the n equations

$$e^{iNk_j} = (-1)^{n-1} \prod_{m=1, m \neq j}^{n} t_{m,j}/t_{j,m} \tag{45}$$

for $j = 1,..., n$. Together with (44), this implies (16).

We remind the reader of Eq. (14), namely

$$s_{ij} = 1 - 2\Delta e^{ik_i - 2H} + e^{i(k_i + k_j) - 4H} \tag{46}$$

We refer to (44)–(46) as "Bethe's equations," which is a slight extension of the terminology of Fabricius and McCoy [15, Eq. (1.2)]. They are *sufficient* conditions for (7) and (8) to have a non-zero solution for the coefficients $A(p_1,..., p_n)$. They form a set of *coupled* equations for the e^{ik_j}, s_{ij} and the ratios $t_{ij} : t_{ji}$.

Apart from Section 6, the problems we shall be discussing occur when some of these variables are zero or infinite. The resolution is always to re-express (5) and (43)–(45) in terms of finite combinations of powers of the variables e^{ik_j} and t_{ij}, to solve the equations simultaneously for these, and to allow for the possibility of a solution containing one or more arbitrary degrees of freedom.

Momentum

When $v = -\lambda$, $e^{(2n-N)H} T(v)/c^N$ is the matrix with entries $\prod_i \delta(\sigma_i, \sigma'_{i+1})$. It shifts all arrows one column to the right. Doing this to the eigenvector g is equivalent to multiplying g by $\exp[i(k_1 + \cdots + k_n)]$. Similarly, the matrix $e^{(N-2n)H} T(\lambda)/c^N$ shifts all arrows one to the left. Hence, writing Λ as $\Lambda(v)$,

$$e^{(2n-N)H} \Lambda(-\lambda)/c^N = c^N e^{(2n-N)H}/\Lambda(\lambda) = e^{i(k_1 + \cdots + k_n)}$$

and $e^{i(k_1 + \cdots + k_n)}$ must be an Nth root of unity, in agreement with (17). From (23) it follows that

$$e^{i(k_1 + \cdots + k_n)} = e^{2nH} Q(\lambda)/Q(-\lambda) \tag{47}$$

Numerical Calculations

Method

To fix our ideas, we conducted a number of numerical experiments on the above equations for lattices of small size: up to $N = 8$. We fixed H, λ and ρ and evaluated the matrix coefficients T_r in (27). We then assigned v an arbitrary value and diagonalized $T(v)$ directly to obtain the eigenvector matrix P, and verified that it did indeed diagonalize all the T_r. For each eigenvalue Λ this gave us the coefficients t_r in (29). We constructed the matrix **B** and determined its null space, and hence all solutions of (41) for the q_j. This gave us the function $Q(v)$. We calculated its zeros to obtain the v_j from (22). We then calculated the κ_j and the wave numbers k_j from (20), and then the s_{ij} from (46). Usually the s_{ij} were *all* non-zero and we took $t_{ij} = s_{ij}$, and calculated the coefficients $A(p_1,\ldots, p_N)$ from (43) and the elements of g from (5). Finally we normalized this vector (so its largest element was one) and compared it with the correspondingly normalized column of the matrix P.

Results for $H \neq 0$

We first took H to be non-zero, either real or pure imaginary, and λ to also be either real or pure imaginary. At this stage we avoided the "roots of unity" cases when $i\lambda$ is a rational fraction of π. We encountered no problems with the above procedure. For a given number n of down arrows, we found no eigenvalues of $T(v)$ that were identically degenerate for all v.[9] The column nullity of **B** was always 1, so there was only one solution (to within normalization) of (41) for the q_j. The s_{ij} were all non-zero and the wave numbers k_i were all distinct so that (5) gave a unique non-zero eigenvector. We worked to about 17 decimal digits of accuracy, and the error in the eigenvector elements was no bigger than 10^{-13}.

For H pure imaginary, we also observed that the eigenvector matrix P was unitary, that for every v_j there was a conjugate according to the rule given after Eq. (28), and that the wave numbers $k_1,\ldots, k_n$ were either real or occurred in complex conjugate pairs. The functions $Q_1(v)$, $Q_2(v)$ had n, $N-n$ finite zeros respectively, and satisfied (37).

3. PARTICULAR VALUES OF *H, Λ*: SOME VERY SPECIAL EIGENVECTORS

Here we consider the case when H, λ satisfy the relation

$$(-e^{\lambda\pm 2H})^N = 1 \tag{48}$$

[9] Obviously they can be degenerate for special values of v: if $v = \pm\lambda$, then $c^{-N}T(v)$ is the "momentum" operator that shifts all arrows in a row one column to the right (or one to the left) and has eigenvalues which are Nth roots of unity.

and show that the Bethe ansatz then admits some some very special eigenvectors. They are the analogues of the special eigenvectors of the zero-field eight-vertex model obtained by the author in Section 7 of ref. 31.

Almost the simplest ansatz one can imagine for an eigenvector ψ of the 2^N by 2^N transfer matrix $T(v)$ is the direct product form

$$\psi = \begin{pmatrix} 1 \\ g_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ g_2 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ g_N \end{pmatrix} \tag{49}$$

where $g_1, \ldots, g_N$ are some parameters to be determined.

Define q by (24): $q = -e^{\lambda}$, and let

$$\tilde{q} = q^{\pm 1} \tag{50}$$

making one of the two possible sign choices here and in the following equations.

Following the method of Section 9.8 of ref. 27, we find that $T(v)\,\psi$ has a simple structure if

$$g_j = (e^{2H}\tilde{q})^j\, g \qquad \text{for} \quad j = 1, \ldots, N \tag{51}$$

except that we need the cyclic boundary condition $g_{N+1} = g_1$: this implies

$$(e^{2H}\tilde{q})^N = 1 \tag{52}$$

which is (48). The parameter g is arbitrary: we can choose it at will. Hence we can regard ψ as a function $\psi(g)$ of g.

Then we find that

$$T(v)\,\psi(g) = \omega_1^N \psi(g') + \omega_4^N \psi(g'') \tag{53}$$

where

$$g' = \tilde{q} g, \qquad g'' = \tilde{q}^{-1} g \tag{54}$$

Now look at the sub-space with n down arrows and $S_z = N - 2n$. Then $\psi(g) = g^n \phi_n$, where, in terms of the positions $x_1, \ldots, x_n$ of the down arrows, ϕ_n is a vector with entries

$$\phi_n(x_1, \ldots, x_n) = (\tilde{q})^{(x_1 + \cdots + x_n)} \tag{55}$$

In this sub-space (53) becomes

$$T(v)\,\phi_n = (a^N e^{NH} \tilde{q}^n + b^N e^{-NH} \tilde{q}^{-n})\,\phi_n \tag{56}$$

Thus ϕ_n is an eigenvector of $T(v)$, with eigenvalue

$$\Lambda = a^N e^{NH} \tilde{q}^n + b^N e^{-NH} \tilde{q}^{-n} \tag{57}$$

This is true for all n from 0 to N, provided only that the restriction (48), or more specifically (52), is satisfied.

Reconciliation with the Bethe Ansatz

How can we reconcile this with the Bethe ansatz? Simply by taking

$$v_j \to \mp\infty \qquad \text{for} \quad j = 1,\ldots, n \tag{58}$$

Then, from (20),

$$e^{ik_j} \to e^{2H} \tilde{q} \qquad \text{for} \quad j = 1,\ldots, n \tag{59}$$

so all the exponential factors containing $x_1,\ldots, x_n$ in (5) are proportional to $\phi_n(x_1,\ldots, x_n)$ and

$$g \propto \phi_n \tag{60}$$

Also, from (22) and (23), we obtain the result (57) for the eigenvalue Λ.

It would seem that there is nothing more to say: the vector ϕ_n is a special case of the Bethe ansatz when all the v_j tend to $\mp\infty$. The eigenvalue is given by (23).

However, the alert reader will notice that we have said nothing about Bethe's equations. More seriously, all the k_j are finite and equal. This is a potential problem in itself, since a casual inspection of (44) suggests that it implies that $t_{i,j} = t_{j,i}$ for all i, j from 1 to n. If we then take $A(P)$ to be given by (43) and substitute into (5), all terms will be equal except for the sign factor ϵ_P. They will therefore all cancel and we shall obtain $g = 0$. (We return to the general problem of what happens when $k_i = k_j$, $s_{i,j} = s_{j,i} \neq 0$ for some pair of values i, j, in Section 6.)

However, $e^{i\kappa_j} = \tilde{q}$, so from (14) and (24),

$$s_{j,m} = 0 \qquad \text{for all} \quad j, m \tag{61}$$

At first sight this appears to only make matters worse. If we use the "normal" solution of (44), namely $t_{j,m} = s_{j,m}$, then (43) gives $A(P) = 0$. Now each term in (5) vanishes, not just their sum!

The answer is of course that one should not use this solution. In fact there is now no reason to use the subsidiary "ansatz" (43) at all. For any finite choice of coefficients $A(P)$, Eqs. (7) are satisfied simply because the $s_{j,m}$ vanish. All that remains is to satisfy (8). One simple choice that does this is to take

$$A(P) = 1 \tag{62}$$

for all permutations P. From (52) and (59),

$$e^{iNk_j} = 1 \tag{63}$$

so both (7) and (8) are satisfied and (5) becomes

$$g(x_1, \ldots x_n) = n!\, e^{ik(x_1 + \cdots + x_n)} \tag{64}$$

where $k = k_1 = \cdots = k_n$. Apart from the non-zero normalization factor $n!$, this is the result (55).

These special eigenvectors are a good illustration of how one can satisfy the original Bethe ansatz equations when some of the $s_{i,j}$ vanish. *The form (43) is not part of the original ansatz, but an addition to it*. It is a necessary consequence of (7) if all of the $s_{i,j}$ are non-zero, but if enough of them vanish it ceases to be necessary.

Still, if (43) is true in general we should not be too ready to abandon it in particular. We can still take $A(P)$ to be given by (43). Equation (44) now imposes no restriction on the $t_{i,j}$, but (45) does. For the case that we are discussing in this section, a simple solution of (45) that avoids the problems that occur when $t_{i,j} = t_{j,i}$ is to take $t_{i,j} = -t_{j,i} \neq 0$ for all i, j. Then $A(P)$ is independent of P and we regain the solution just discussed.

Another solution can be obtained, under the more specialized conditions (92) and (93), by letting the v_j in Section 5, for the case $M = n$, tend to $\mp\infty$.

More generally, we can take the t_{ij} to be arbitrary and non-zero, with $t_{ij} \neq t_{ji}$, for $1 \leqslant i, j \leqslant n-1$, and then use (45) to determine the ratios $t_{j,n}/t_{n,j}$.

In all three approaches, note that we use (45), rather than (44), to determine some of the ratios t_{ij}/t_{ji}. This is a basic feature of the algebra of the next two sections.

Infinite v_js: The General Situation

Suppose just r of the $v_1, \ldots, v_n$ equal $-\infty$, s equal $+\infty$, and the remaining $n-r-s$ are finite and arbitrary. More precisely:

$$v_j = \text{finite}, \qquad \text{for} \quad j = 1, \ldots, n-r-s$$

$$v_j = -\infty, \qquad e^{i\kappa_j} = q, \qquad \text{for} \quad j = n-r-s+1, \ldots, n-s \tag{65}$$

$$v_j = +\infty, \qquad e^{i\kappa_j} = q^{-1}, \qquad \text{for} \quad j = n-s+1, \ldots, n \tag{66}$$

Let us call these three cases type 1 to type 3, respectively. From (14) and (24), if v_j is of type 2 and v_m is not of type 2, then

$$s_{jm} = (q - e^{ik_m})/q, \qquad s_{mj} = -q(q - e^{ik_m}) \tag{67}$$

so $s_{mj}/s_{jm} = t_{mj}/t_{jm} = -q^2$. Similarly, if v_j is of type 3 and v_m is not of type 3, then $s_{mj}/s_{jm} = t_{mj}/t_{jm} = -q^{-2}$.

If v_j and v_m are both of type 2, or both of type 3, then $s_{jm} = s_{mj} = 0$ and $A(P)$ is not necessarily given by (43). Strictly, we should go back to the original Bethe ansatz equations (7) and (8).

However, for similar reasons to those above, it seems that we can without loss of generality take $A(P)$ to be given by (43), so long as we realise that (44) no longer defines t_{jm}/t_{mj} for v_j and v_m both of type 2, or both of type 3. Then (7) gives Eq. (45). Taking v_j to be of type 1, 2 or 3, we obtain

$$\begin{aligned}
e^{iNk_j} &= q^{2s-2r} \prod_{m=1}^{n-r-s} (-s_{m,j}/s_{j,m}), \qquad j = 1, \ldots, n-r-s \\
e^{2NH} q^N &= q^{2n-2r} \prod_{m=n-r-s+1}^{n-s} (-t_{m,j}/t_{j,m}), \qquad j = n-r-s+1, \ldots, n-s \\
e^{2NH} q^{-N} &= q^{2s-2n} \prod_{m=n-s+1}^{n} (-t_{m,j}/t_{j,m}), \qquad j = n-s+1, \ldots, n
\end{aligned} \tag{68}$$

where all three products exclude the value $m = j$.

Taking the product of each of these three equations over the allowed values of j, the $-s_{m,j}/s_{j,m}$, $-t_{m,j}/t_{j,m}$ factors cancel out, leaving

$$\begin{gathered}
e^{iN(k_1 + \cdots + k_{n-r-s})} = q^{2(s-r)(n-r-s)} \\
e^{2NrH} = q^{r(2n-2r-N)}, \qquad e^{2NsH} = q^{s(N+2s-2n)}
\end{gathered} \tag{69}$$

Eliminating H gives

$$q^{2rs(N+r+s-2n)} = 1 \tag{70}$$

so we see that such infinite zeros can only occur when r, s or $N+r+s-2n$ vanishes, or when q (and e^{2H}) is a root of unity.

Let $\hat{Q}(v)$ be given by (22), but with the product restricted to the finite v_j:

$$\hat{Q}(v) = \prod_{j=1}^{n-r-s} \sinh[(v-v_j)/2] \tag{71}$$

then, to within factors independent of v,

$$Q(v) = e^{(r-s)\,v/2}\, \hat{Q}(v) \tag{72}$$

Substituting into the eigenvalue equation (23), we obtain

$$\Lambda = (-1)^n \frac{\omega\phi(\lambda-v)\,\hat{Q}(v+2\lambda) + \omega^{-1}\,\phi(\lambda+v)\,\hat{Q}(v-2\lambda)}{\hat{Q}(v)} \tag{73}$$

where

$$\omega = e^{NH}(-q)^{r-s} \tag{74}$$

Hence

$$\omega^{2r} = q^{r(2n-2s-N)}, \qquad \omega^{2s} = q^{s(N+2r-2n)}$$

and we see that if r, s, $N+r+s-2n$ are all non-zero, then ω must be a root of unity.

4. THE ZERO-FIELD MODEL: $H=0$

The next step was to turn off the field, setting $H=0$. For N odd no problems appeared: we were able to calculate all the eigenvectors in all the subspaces $n=0,\ldots, N$ without difficulty. For N even we encountered two problems, both of which have been discussed previously in the literature.

"Beyond the Equator": N Even and n > N/2. Q(v) Has Infinite Zeros

In this "beyond the equator" case,[10] (37) has the simple solution

$$Q_2(v) = Q_1(v) \tag{75}$$

Certainly (41) permits this solution, and we observe numerically that **B** has nullity one, so it is the only solution. Hence $Q(v)$ is the same for n as for $N-n$. This is quite consistent with (22)—it merely means that $2n-N$ of the zeros have gone off to infinity. This is even easier to see in (40): the first $n-N/2$ coefficients q_j vanish, as do the last $n-N/2$ coefficients. The degree of the Laurent polynomial is reduced from n to $N-n$, which is still consistent with (41).

From (20) and (24), this means that $n-N/2$ of the wave numbers $k_1,..., k_n$ are given by $e^{ik_j}=q$, and another $n-N/2$ by $e^{ik_j}=1/q$.

Faddeev and Takhtajan[(9)] state that "Bethe's vector vanishes" for $n>N/2$. For $n=1+N/2$ this is certainly not true of the vector g as given by (5) and (43). We have observed numerically that it is non-zero and that it is indeed the eigenvector of T.

For $n>1+N/2$ there is a problem, but it can be overcome, using the working at the end of the last section, taking r, s therein to be $n-N/2$, so that (69) is satisfied for all λ, q. When $v_j=v_m=\pm\infty$, then $s_{m,j}=s_{j,m}=0$, so (44) tells us nothing about $t_{m,j}$, $t_{j,m}$: instead they should be chosen to satisfy the last two equations of (68). There will be many ways to do this, corresponding to the fact that $A(P)$ enters (5) only via its sum over all ways of permuting $v_{N-n+1},..., v_{N/2}$ and $v_{N/2+1},..., v_n$. One simple way is to take $t_{m,j}=-t_{j,m}$. The $N-n$ finite zeros v_j are given by the first of the equations (68), which is the same as (45) when n is replaced by $N-n$, i.e., the Bethe equations for the *right* side of the equator. The eigenvalue equations are the same for both n and $N-n$.

The eigenvector equations are different, since we must include all n zeros in the product in (43). The resulting coefficients $A(p_1,..., p_n)$ are finite and non-zero. The Eqs. (7) and (8) are satisfied, so the vector g with elements (5) must be an eigenvector if it is non-zero. It is non-zero in our numerical experiments, and it is the eigenvector corresponding to the eigenvalue Λ.

Thus despite the assertions that have been made in the past, the Bethe ansatz can be used to construct the eigenvector g for $n>N/2$, even when N is even. It is a furphy that Bethe's ansatz does not work on the wrong side of the equator.

N Even and $2\leqslant n\leqslant N-2$: A Single Bound Pair

The other problem that we encountered first occurs for $N=4$ and $n=2$, then for even N and $2\leqslant n\leqslant N-2$. It is referred to by Bethe himself [1, after Eq. (23)] and has been considered by others since [11, 13, 34, Eq. (3.2.23b)]. For some eigenvalues with momentum ± 1, i.e., $k_1+\cdots+k_n=0$ or π, we found that $Q(v)$ had a pair of zeros v_1, v_2 such that $v_1=\lambda$, $v_2=-\lambda$. From (20) this implies that

$$e^{ik_1}=e^{-ik_2}=0 \tag{76}$$

More strongly, it was always true for such pairs that

$$e^{i(k_1+k_2)} = -1 \tag{77}$$

so from (46),

$$s_{12} = 2\Delta e^{-ik_1} = \infty \tag{78}$$

while s_{21} vanishes.

Since s_{12}, s_{21} do not both vanish, we can take as usual $t_{12} = s_{12}$, $t_{21} = s_{21}$.

In fact t_{21} vanishes strongly as this situation is approached (say by turning off the field H): from (45),

$$t_{21} = C_{21}\, e^{i(N-1)\,k_1} \tag{79}$$

where C_{21} has a finite non-zero value given by (45).

There is no problem solving Bethe's equations for the other $k_3,\ldots, k_n$ and for C_{21}. In principle one can then substitute these expressions into (43) and (5) and extract the terms and elements that grow most rapidly as e^{ik_1} vanishes. Choosing C as stated after (44), each term in the summand of (5) will give a finite contribution to the eigenvector g. Some will be zero, but our numerical experiments indicate that the total vector g is not zero, and is in fact the correct eigenvector.

Numerical Results

For our numerical experiments, we simply assigned e^{ik_1} the numerically small but non-zero value $10^{-14}i$ (the i being necessary for k_1, k_2 to be complex conjugates), calculated the s_{ij} other than s_{21} from (46), then calculated s_{21} from (45), substituted the results into (43) and (5) and normalized the vector g. We obtained the correct eigenvector to approximately 14-digit accuracy.

For $N=4$ there was just one such eigenvalue Λ, in the $n=2$ central block. For $N=6$ there was one in the $n=2$ block, two in the $n=3$ block, and one in the $n=4$ block. For $N=8$ there were $1, 2, 5, 2, 1$ in the $n=2, 3, 4, 5, 6$ blocks, respectively. This suggests (tentatively) that the Catalan numbers may count such eigenvalues. The momenta were -1, except for a single eigenvalue with momentum $+1$ in each block with $3 \leqslant n \leqslant N-3$.

When $N/2+1 < n < N-1$, there are eigenvalues where the two problems we have just discussed occur together, i.e., more than one of the

e^{ik_j} are equal to q, more than one to $1/q$, and two of the others are $\pm i\infty$. The above procedures were built into our computer program and worked perfectly, giving the correct non-zero eigenvector of T.

For N even, we also calculated the matrix $\Phi_n(v)$ of Eq. (96) of ref. 31. We found that its rows were, as expected, linearly independent, so we were able to use (32) above to calculate $\tilde{Q}(v)$, and did indeed find that this matrix was diagonalized by the same matrix P that diagonalizes $T(v)$, and that each eigenvalue was the function $Q(v)$ discussed above (more precisely, the eigenvalue was $Q(v)/Q(v_0)$).

The main lesson from this and the previous section is that Bethe's equations should be viewed as a set of coupled non-linear equations for the e^{ik_j}, s_{ij} and the ratios $t_{ij} : t_{ji}$. We do not necessarily proceed by solving (46) and (44) for the ratios $t_{ij} : t_{ji}$. For some i and j it may be appropriate to obtain this ratio from (45).

5. $H=0$ AND q A ROOT OF UNITY: EXACT COMPLETE STRINGS

Now we come to the case discussed by Deguchi, Fabricius and McCoy in refs. 14–17, where λ is a rational fraction of $i\pi$, i.e., there exist integers ν, M (with no common factors) such that

$$\lambda = i\nu\pi/M, \qquad q = -e^{i\nu\pi/M} \tag{80}$$

using (24).

Then

$$q^{2M} = 1 \tag{81}$$

and there is no smaller integer power of q^2 that equals one.

For the moment we allow H to be arbitrary: we shall show that the further restriction (91) is necessary for a string (more precisely, a single string) to occur, and from then on take H to be zero.

The set of Bethe zeros $v_1,\ldots, v_n$ may now contain one or more "complete strings," in which M of them, say $v_1,\ldots, v_M$, are related by

$$v_j = v_1 + 2(j-1)\,\lambda \qquad \text{for} \quad j = 1,\ldots, M \tag{82}$$

This implies that $v_{j+1} = v_j + 2\lambda$ and $v_1 = v_M + 2\lambda - 2i\nu\pi$, so from (21),

$$s_{12} = s_{23} = \cdots = s_{M-1,M} = s_{M,1} = 0 \tag{83}$$

We see that some of the s_{ij} vanish, so we have to be careful with Bethe's equations.

The set $\{v_1,\ldots, v_M\}$ may of course contain more than one complete string. For simplicity, from now on we shall restrict our attention to the case when there is only one string, but we fully expect our methods and comments to be applicable to the general case.

For $j = 1,\ldots, M$, Eq. (16) becomes $0 = 0$, which is true, but not helpful. For $j = M+1,\ldots, n$, both sides of the equation are non-zero. Using the form (21) of s_{ij}, we obtain

$$s_{1,j}s_{2,j}\cdots s_{M,j} = (-1)^M\, s_{j,1}s_{j,2}\cdots s_{j,M} \tag{84}$$

so for $j > M$, Eq. (16) simplifies to

$$e^{iNk_j} = (-1)^{n-1-M} \prod_{m=M+1,\, m\neq j}^{n} s_{m,j}/s_{j,m} \tag{85}$$

Taking the product of these $n-M$ equations, the $s_{j,m}$, $s_{m,j}$ factors cancel, leaving

$$e^{iN(k_{M+1}+\cdots+k_n)} = 1 \tag{86}$$

From (17) it follows that

$$e^{iN(k_1+\cdots+k_M)} = 1 \tag{87}$$

Now we note from (82) and (20) that, for $j = 1,\ldots, M$,

$$e^{ik_j} = qe^{2H}\,\frac{1+q^{2j-3}z_1}{1+q^{2j-1}z_1} \tag{88}$$

where $z_1 = \exp(v_1)$ and more generally

$$z_j = e^{v_j} = q^{2j-2}z_1 \tag{89}$$

Substituting this into (87) and using (81), we obtain

$$q^{NM}e^{2NMH} = 1 \tag{90}$$

This is an extra condition on q and H that must be satisfied for a complete string to occur.[10] From this and (81) we see that

$$e^{4NMH} = 1 \tag{91}$$

[10] At least for just one string to occur, but the same condition appears to be necessary for any number of strings.

An obvious and interesting solution of this equation is

$$H = 0 \tag{92}$$

and from now on in this section we shall take $H = 0$, but we do note that there are other (pure imaginary) values of H for which complete strings may occur.

If N is even, (90) is implied by (81). If N is odd, the two equations together imply that $q^M = 1$, but this is consistent with q^2 being a primitive Mth root of unity only if M is odd. Thus we have two possibilities:

$$\begin{aligned} q^{2M} &= 1, \qquad N \text{ even} \\ q^M &= 1, \qquad N \text{ and } M \text{ both odd} \end{aligned} \tag{93}$$

Apparent Difficulties: (1) Calculating g

There are two problems that appear when a complete string occurs: the Bethe equations do not have a unique solution, and if we use the obvious solution $t_{ij} = s_{ij}$ of (44) in (43), then every coefficient $A(P)$ vanishes, so the eigenvector g, given by (5) also vanishes.[11]

Let us dispose of the second difficulty first, since it is fairly straightforward. For the algebraic Bethe ansatz, it is considered by Fabricius and McCoy in the remarks after their equation (1.36) of ref. 17.

We can still use the ansatz (43) for the coefficients $A(P)$ and attempt to satisfy the modified Bethe's equations (44), (45). Taking $t_{ij} = s_{ij}$ for all i, j, we note from (83) that

$$t_{12} = t_{23} = \cdots = t_{M,1} = 0$$

For $j = M+1,\ldots, n$, (45) becomes the above reduced equations (85), which can be viewed as fixing $v_{M+1},\ldots, v_n$.

Let v_1 be assigned arbitrarily. Then $v_2,\ldots, v_M$ and $e^{ik_1},\ldots, e^{ik_M}$ are given by (82) and (88).

We still have to satisfy (45) for $j = 1,\ldots, M$. We can do this by using it to determine the ratios $t_{j-1,j}/t_{j,j+1}$, for $j = 1,\ldots, m$ (with $t_{0,1} = t_{m,m+1} = t_{m,1}$). We could of course have originally formulated Bethe's equations (which are just a set of algebraic equations and may well have solutions at zero or infinity) in terms of these non-zero, finite ratios.

[11] The only way the rhs of (43) could be non-zero would be for it to contain the factors $s_{21}, s_{32},\ldots, s_{M,M-1}, s_{1,M}$. But this cannot happen as there is no inverse permutation P' such that $p'_2 > p'_1, p'_3 > p'_2,\ldots, p'_1 > p'_M$: the inequalities are inconsistent. At least one of them must fail, which is the reason for the renormalization of $A(P)$ proposed below.

We also take C in (43) to be t_{12}. Then some of the coefficients $A(P)$ will depend on $t_{12}, t_{23},\ldots, t_{M,1}$ only via their ratios, which are given by (45). The remaining coefficents $A(P)$ will vanish.

Thus all the coefficients are finite, some are non-zero, and we may hope that the eigenvector g, given by (5), will befinite and non-zero. In our numerical experiments this is what we have found.

If there are γ complete strings, then C should be the product of γ factors t_{ij}, being one of the vanishing t_{ij} from each string.

Apparent Difficulties: (2) Non-Uniqueness of the Eigenvector

We said above "let v_1 be assigned arbitrarily." Why is this allowed? Should not its value be determined? This is the problem that concerned Deguchi, Fabricius and McCoy.[14–17]

The answer is that for any value of v_1 the above procedure satisfies (43)–(45) and therefore the Bethe ansatz equations (5)–(17). Provided g is not zero (and our numerical calculations indicate that it is not), then it must be an eigenvector of the transfer matrix $T(v)$. We are free to choose v_1 as we wish. There is no *a priory* need to "complete" Bethe's equations.[16]

Let us look more closely at what is happening.

From (82) and (22), the M zeros $v_1,\ldots, v_M$ contribute to $Q(v)$ a factor

$$\prod_{j=1}^{M} \sinh[(v-v_j)/2] \propto \sinh[M(v-v_1)/2] \tag{94}$$

This factor cancels out of (23), except only for a constant $(-1)^{\nu}$.

It follows at once that if $Q(v)$ satisfies (23), and hence the Bethe equations, *then so will any other function $Q(v)$ with a different value of v_1.*

There is nothing remarkable about this. It means that the matrix $\mathbf{B}$ has rank less than n, so there is more than one solution $q_0,\ldots, q_n$ of (41). This in turn is a signal that the eigenvalue $\Lambda(v)$ is degenerate, for *all* v.

Of course, changing v_1 will change $Q(v)$, so the matrix $\tilde{Q}(v)$ will not be degenerate. If we construct it explicitly as above, then diagonalizing $\tilde{Q}(v)$ rather than $T(v)$ will resolve the degeneracies of $T(v)$. Further, these will give the eigenvalues and eigenvectors obtained by taking the limit as λ approaches the value (80). This is what Fabricius and McCoy have done, and the results are interesting.

What we are concerned with here is showing that there is nothing in their work that indicates that the Bethe ansatz is incomplete, in the usual sense of not giving all the eigenvectors or "states." The fact that $Q(v)$ is not uniquely defined by (5)–(17) is precisely the reason why one can use these equations to obtain a complete basis of the eigenspace of the eigenvalue $\Lambda(v)$.

A Single String Containing all the $v_1,\dots, v_n$

For simplicity we further restrict our attention to the case when *all* the $v_1,\dots, v_n$ lie in the string, i.e.,

$$M = n \tag{95}$$

However, we expect the substance of our remarks to generalize to $n > M$, and indeed to the case when $v_1,\dots, v_n$ contain more than one string.

The eigenvalue Λ is given immediately by (23):

$$\Lambda = q^n[\phi(\lambda - v) + \phi(\lambda + v)] = q^n(a^N + b^N) \tag{96}$$

To within a sign, it is the eigenvalue for the "vacuum" state, when all spins are up. We shall now use the original Bethe ansatz equations (5)–(17) to obtain explicit expressions for all the eigenvectors g corresponding to this eigenvalue, for a given value of the number n of down arrows. We shall *not* use "Bethe's equations" (43)–(45).

From (82) and (21),

$$s_{12} = s_{23} = \cdots = s_{n-1,\,n} = s_{n,\,1} = 0 \tag{97}$$

The coefficients $A(1, 2,\dots, n)$, $A(2, 3,\dots, n, 1),\dots$, $A(n, 1, 2,\dots, n-1)$ enter Eqs. (7) only with multiplying factors s_{ij} that belong to the set (98). This means that these coefficients do not enter at all. It appears that all other coefficients do enter with non-zero multiplying factors, and the equations ensure that they vanish.[12]

Choosing $A(1, 2,\dots, n) = 1$, from (8) it follows that

$$A(j, j+1,\dots, n, 1,\dots, j-1) = e^{iN(k_j + k_{j+1} + \cdots + k_n)} \tag{98}$$

From (5) it follows that

$$\begin{aligned} g(x_1,\dots, x_n) = \sum_{j=1}^{n} \exp\{&i[k_1 x_{n+2-j} + k_2 x_{n+3-j} + \cdots + k_{j-1} x_n + k_j(x_1 + N) \\ &+ k_{j+1}(x_2 + N) + \cdots + k_n(x_{n+1-j} + N)]\} \end{aligned} \tag{99}$$

The cyclic property $g(x_1,\dots, x_n) = g(x_2,\dots, x_n, x_1 + N)$ is manifested by (99).

All of the Bethe ansatz equations (5)–(15) are now satisfied. We still have the parameter v_1, or equivalently z_1, at our disposal.

[12] I have only verified this for $n = 2,\dots, 9$, but this strongly suggests that it is correct.

Substituting (88) into (99) and dividing by a common normalisation factor $(1-(-z_1/q)^n)/n(1+z_1/q)^N$, we obtain

$$g(z_1 \mid x_1, \ldots, x_n) = n^{-1} \sum_{j=1}^{n} q^{x_1 + \cdots x_n + N(1-j)} \prod_{r=1}^{n} (1 + q^{2j+2r-3} z_1)^{x_{r+1} - x_r - 1} \tag{100}$$

taking $x_{n+1} = x_1 + N$ and exhibiting the dependence of g on z_1.

From (93), $q^{Nn} = 1$, which means that the summand in (100) is unchanged by replacing j by $j+n$. Replacing j, z_1 by $j-1$, $q^2 z_1$, we observe that

$$g(x^2 z_1 \mid x_1, \ldots, x_n) = q^N g(z_1 \mid x_1, \ldots, x_n) \tag{101}$$

Because of the restrictions (6), the RHS of (100) is a polynomial in z_1 of degree $N-n$. Hence we can expand the vector g and its elements in powers of z_1:

$$\begin{aligned} g(z_1) &= \sum_{k=0}^{N-n} z_1^k c_k \\ g(z_1 \mid x_1, \ldots, x_n) &= \sum_{k=0}^{N-n} z_1^k c_k(x_1, \ldots, x_n) \end{aligned} \tag{102}$$

c_k being a vector with elements $c_k(x_1, \ldots, x_n)$.

Substituting the expression (102) into (101), we find that

$$c_k = 0 \qquad \text{unless} \quad q^{2k} = q^N \tag{103}$$

This means that most of the c_k vanish. Define an integer α by

$$\begin{aligned} \alpha &= N/2, & &\mod n & &\text{if } N \text{ is even} \\ \alpha &= (N-n)/2, & &\mod n & &\text{if } N, n \text{ are odd} \end{aligned} \tag{104}$$

so that in either case $0 \leqslant \alpha < n$. Then c_k is non-zero only when $k = \alpha$, $\alpha + n$, $\alpha + 2n, \ldots$. It follows that there are at most

$$\mathcal{N} = \left[\frac{N-\alpha}{n} \right] \tag{105}$$

non-zero vectors c_k in the expansion (102). Here $[x]$ denotes the integer part of x.

We can write down an explicit, if unwieldy, expression for the elements of c_k by performing a binomial expansion on each product in (100):

$$c_k(x_1,\dots,x_n) = q^{x_1+\cdots+x_n} \sum_{m_1,\dots,m_n} \prod_{r=1}^{n} q^{(2r-1)\,m_r} \binom{x_{r+1}-x_r-1}{m_r} \tag{106}$$

where k must satisfy the restriction (103) and the summation is over all integers $m_1,\dots, m_n$ such that $m_1+\cdots+m_n=k$ and

$$0 \leqslant m_r \leqslant x_{r+1}-x_r-1, \qquad \text{for} \quad r=1,\dots, n$$

Numerical Tests

It is not obvious whether these vector are in fact linearly independent. The author knows of no reason to suppose they are not, but as a check we have numerically calculated the vectors for $n=2$ and 3. We can distinguish three cases:

(i) $n=2$, $\lambda=i\pi/2$, $q=-i$, $q^n=-1$,
(ii) $n=3$, $\lambda=i\pi/3$, $q=e^{-2\pi i/3}$, $q^n=1$,
(iii) $n=3$, $\lambda=2i\pi/3$, $q=e^{-\pi i/3}$, $q^n=-1$.

For all these cases $q^{2n}=1$, but only for the second is $q^n=1$. Thus N must be even for cases (i) and (iii), but may be either even or odd for case (ii).

We present the results in Table I, for $N=2,\dots, 16$. In every case we calculated the vector g with elements (100) for 12 randomly chosen values of z_1 and then determined (to 15 digit precision) the rank $\tilde{r}$ of the matrix with these 12 column vectors. We then numerically verified that each was an eigenvector of the six-vertex model transfer matrix T, with eigenvalue (96). Finally, we calculated the column nullity $\tilde{n}$ of $T-\Lambda I$ in the subspace with n down arrows. This is the degeneracy of Λ. In every case we found

Table I. The Dimensions of the Space $\mathscr{V}$ for $n=2$ and $n=3$, as Calculated Numerically (a Value of Zero Implies that Their Are No Eigenvectors and that (24) Is Not an Eigenvalue). In Every Case the Result Agrees with (106) and with the Calculated Nullity of $T-\Lambda I$, Implying that the Vectors (107) Are Indeed Linearly Independent, and that $\mathscr{V}$ Is the Complete Eigenspace

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
case (i)	0		2		2		4		4		6		6		8
case (ii)	0	1	0	1	2	1	2	3	2	3	4	3	4	5	4
case (iii)	0		0		2		2		2		4		4		4

$\tilde{r} = \tilde{n} = \mathcal{N}$, where $\mathcal{N}$ is given (105). Thus at least for these cases $\mathcal{V}$ is indeed of dimension (105) and contains all eigenstates.

We can compare these results with those of Fabricius and McCoy.[15] Their γ is related to our λ by $\lambda = i\gamma$, so $\Delta = -\cos\gamma$ and $q = -e^{i\gamma}$. From Table II of their paper, when $\gamma = \pi/2$, $\Delta = 0$, $n = 2$ and $N = 16$, there are eight single strings in the $S^z = N/2 - n = 6$ sub-space, all corresponding to the same eigenvalue. This agrees with the above derivation: $\alpha = 0$, so $\mathcal{N} = 8$ and the summand in (102) is non-zero only when k takes the eight values 0, 2, 4, 6, 8, 10, 12, 14.

Also, from Table VII of the same paper, when $\gamma = \pi/3$, $\Delta = -1/2$, $n = 3$ and $N = 16$, there are four equal-eigenvalue single strings in the $S^z = N/2 - n = 5$ sub-space. This also agrees with the above, and with case (ii) in the table: $\alpha = 2$, $\mathcal{N} = 4$, and $k = 2, 5, 8, 11$.

Some of these results have also been obtained by Braak and Andrei,[35] who refer to the freedom in the choice of the string centres as "transparent excitations." Their Table I is line 2 of our Table I above.

Conclusions

Let $\mathcal{V}$ be the space that spanned by the non-zero vectors c_k. Then as z_1 varies, the eigenvector $g(z_1)$ traces a curve within this space. Each vector c_k can be written as a sum of eigenvectors $g(z_1)$, so is itself an eigenvector. $\mathcal{V}$ is an eigenspace corresponding to the eigenvalue (96).

It appears that the vectors c_k are linearly independent and span the full eigenspace (within the sub-space $S^z = N/2 - n$ of n down arrows). If so, then $\mathcal{V}$ is of dimension $\mathcal{N}$. The eigenvalue has degeneracy $\mathcal{N}$, and we have used the Bethe ansatz to construct the full eigenspace.

The Case When the String Parameters $v_1, \ldots, v_m$ Are Infinite

If q satisfies both the restrictions (93) and

$$q^N = 1 \tag{107}$$

then $k = 0$ and $k = N - n$ in (102) both correspond to non-zero vectors c_k. These c_k are the values of g when $z_1 = 0$ and $z_1 = \infty$, i.e., when $v_1, \ldots, v_m = -\infty$ and $v_1, \ldots, v_m = +\infty$. We have a string of infinite $v_1, \ldots, v_m$.

From (100), we readily find that

$$\begin{aligned} c_0(x_1, \ldots, x_m) &= q^{x_1 + x_2 + \cdots + x_m} \\ c_{N-m}(x_1, \ldots, x_m) &= q^{m-N} q^{-x_1 - x_2 - \cdots - x_m} \end{aligned} \tag{108}$$

These are particular cases of the special vectors reported in Eqs. (16)–(22) of ref. 31, in ref. 36, and in Eq. (55) above.

Multiple Complete Strings: The Function $Q(v)$

We also calculated the null space of the matrix **B** in (42), algebraically using Mathematica. We fixed the values of λ and q as in cases (i), (ii), (iii). We then allowed n (the number of down arrows) in (96) and (29)–(42) to take all values from 0 to N, for $N = 2, 3,\dots, 9$. Since $\Lambda(v)$ is given by (96), we could immediately calculate $t_0,\dots, t_N$ and form **B**. We did indeed find that the column nullity of B was sometimes greater than one. For cases (ii) and (iii), where $-3i\lambda$ is an integer, we found that the nullity was zero unless n was a multiple of 3, meaning that $\Lambda(v)$ was not an eigenvalue. When n was a multiple of 3 the nullity was $(n+3)/3$ and (41) was satisfied provided only that q_j was zero when j is not a multiply of 3. Thus

$$e^{nv/2}Q(v) = \text{arbitrary polynomial in } z^3 \text{ of degree } n/3$$

where $z = e^v$. Whatever the choices of the coefficients of this polynomial, it can be factored into $n/3$ polynomials of degree 1 in z^3, each of which is a complete string.

So $Q(v)$ factors into a product of $n/3$ complete strings of length 3. The "centre" (the average value of the three v_js in the string) of each string is undetermined. Any such function $Q(v)$ satisfies (41).

We found corresponding behaviour for case (i), when $\lambda = i\pi/2$: for n even, $Q(v)$ factors into a product of $n/2$ undetermined complete strings of length 2. There no solutions for n odd.

We have not attempted to generalize the derivation of this section of the eigenvector g to such multiple strings, but presume that it can be done, and that one would find the remarkable binomial pattern of degeneracies reported by Fabricius and McCoy in their Tables II and VII for "maximum $S^z = 8$."[15]

In the $2n = N$ sub-space one can impose the "sum-rule" constraint (151) on $v_1 + \cdots + v_n$. (For non-degenerate eigenvalues, with no strings, this will automatically be satisfied. If there are strings, it is not necessary, but may be convenient, and will still give a complete set of eigenvectors.) If $v_1,\dots, v_n$ contain only one complete string, then this condition can be used to fix its centre. For $n = 2$ or 3, and $N = 2n$, this gives $z_1^n = (-1)^r \exp(-n^2\lambda)$. This r is 0 for states symmetric under arrow reversal, 1 for antisymmetric states. We have verified numerically that the resulting eigenvectors (100) do indeed have this (anti-)symmetry.

6. SOME OF THE $v_1,\dots, v_n$ EQUAL

Suppose that all the $s_{i,j}$ are non-zero. Then the $A(p_1,\dots, p_n)$ are given by (43). Substitute this into (5) and for the moment regard $v_1,\dots, v_n$ as arbitrary parameters and $x_1,\ldots, x_n$ as fixed. The result is an entire anti-symmetric periodic function of $v_1,\ldots, v_n$. It must therefore contain the factor

$$\prod_{1 \leqslant i < j \leqslant n} \sinh[(v_i - v_j)/2] \tag{109}$$

In principle one can divide this factor out (it is the same for all $x_1,\ldots, x_n$, so is just a normalization factor for the vector g). The result is a symmetric function of $v_1,\ldots, v_n$. For instance, if $n=2$ and $N=4$, as in (40) we can write $Q(v)$ as a Laurent polynomial in $z=e^v$:

$$Q(v) = z^{-1}(dz^2 + ez + f) \tag{110}$$

We can then follow this procedure so as to write all the $g(x_1, x_2)$ as multinomials in d, e, f. Define

$$\Delta' = 2\Delta = q + q^{-1}$$

$$\xi_1 = q^{-1}d - e + qf$$

$$\xi_2 = qd - e + q^{-1}f$$

$$\xi_3 = (d+f)\,\Delta' - 2e$$

$$\xi_4 = -4e(d+f) + (d^2 + 6df + f^2 + e^2)\,\Delta' - df\,\Delta'^3$$

$$\xi_5 = 2e(e^2 - 3d^2 - 10df - 3f^2) + (d+f)\,\Delta'(d^2 + 14df + f^2 + 3e^2)$$
$$- e\Delta'^2(2df + e^2) - 3df(d+f)\,\Delta'^3 + def\,\Delta'^4$$

then we find that we can normalize the vector g so that

$$\begin{aligned} g(1,2) &= \xi_1^2\xi_3, & g(1,3) &= \xi_1\xi_4 \\ g(1,4) &= \xi_5, & g(2,3) &= \xi_1\xi_2\xi_3 \\ g(2,4) &= \xi_2\xi_4, & g(3,4) &= \xi_2^2\xi_3 \end{aligned} \tag{111}$$

Each element is a multinomial of degree 3 in the coefficients d, e, f. Such expressions remove the difficulty that occurs when two or more of the v_j become equal: in that case the anti-symmetric factor mentioned above vanishes, as does the right-hand side of (5), but the above expressions do not.

This procedure is equivalent to taking the limit of the ratios of the eigenvector elements $g(x_1,\ldots, x_n)$ in (5) as the v_j approach one another. It also has the conceptual advantage of making it clear that one does not necessarily have to calculate the zeros $v_1,\ldots, v_n$ of $Q(v)$. Instead one can imagine solving the bilinear equations given in (29)–(42) for the coefficients of the Laurent polynomial expansions in e^v of $\Lambda(v)$ and $Q(v)$, then using the results in equations such as the above.

Of course, in practice we do not have useful explicit results for the generalizations of (111) to arbitrary n and N, so for anything other than small n, N one is forced to use (5) and (43) directly. However, from this point of view there is nothing remarkable about two of the v_j coinciding: it merely means that $Q(v)$ has a repeated zero. If all one wants is the eigenvalue $\Lambda(v)$, then the alternate form form (29)–(42) of Bethe's equations can be used directly as written.

7. EXTENSION TO THE EIGHT-VERTEX MODEL

The zero-field eight-vertex model was solved in 1971 by the author[28, 29] by extending the functional relation (31) (with $H=0$) to the eight-vertex model, provided that

$$N = \text{even} \tag{112}$$

This restriction applies throughout this section.

This functional relation method gives the eigenvalues $\Lambda(v)$, but not the eigenvectors. In 1972, while at Stony Brook, the author derived equations for the eigenvectors of the eight-vertex model in a sequence of three papers.[31, 37, 38] The basic technique was to convert the eight-vertex model to an ice-type solid-on-solid model, and then to solve this by an appropriately generalized Bethe ansatz.

Here we use the notation of refs. 31, 37, and 38 and prefix the equations by I, II, III, according to in which of the three papers it appears. Papers II and III consider the case when the parameter η satisfies the "root of unity" condition (I.9), (II.6.8) or (III.1.9), i.e.,

$$L\eta = 2m_1 K + i m_2 K' \tag{113}$$

where L, m_1, m_2 are integers. This is analogous to (80). This restriction is *not* needed in Section 6 of I, because the condition (8) therein is sufficient to ensure the required cyclic boundary condition from column N to column 1.

Papers II and III further consider the case when there are integers $\tilde{n}$, v' such that

$$N-2\tilde{n}=Lv' \tag{114}$$

Modified elliptic theta functions are introduced:

$$\begin{aligned} H(u) &= H_{Jb}(u)\exp[i\pi m_2(u-K)^2/(4KL\eta)] \\ \Theta(u) &= \Theta_{Jb}(u)\exp[i\pi m_2(u-K)^2/(4KL\eta)] \end{aligned} \tag{115}$$

$H_{Jb}(u)$, $\Theta_{Jb}(u)$ being the usual Jacobi theta functions [Eq. (15.1.5) of ref. 27]. These modified functions are periodic of period $2L\eta$. The zero-field eight-vertex model Boltzmann weights are then given by (I.8) and (II.6.1):

$$\begin{aligned} a &= \rho\Theta(-2\eta)\,\Theta(\eta-v)\,H(\eta+v) \\ b &= -\rho\Theta(-2\eta)\,H(\eta-v)\,\Theta(\eta+v) \\ c &= -\rho H(-2\eta)\,\Theta(\eta-v)\,\Theta(\eta+v) \\ d &= \rho H(-2\eta)\,H(\eta-v)\,H(\eta+v) \end{aligned} \tag{116}$$

One also uses the functions

$$h(u)=-h(-u)=H(u)\,\Theta(-u) \tag{117}$$

$$\phi(u)=[\rho\Theta(0)\,h(u)]^N \tag{118}$$

which satisfy

$$\begin{aligned} h(u+L\eta) &= (-1)^{m_1(m_2+1)}\,h(u) \\ \phi(-u) &= \phi(u) \end{aligned} \tag{119}$$

$$\begin{aligned} h(u+iK') &= -e^{-i\pi m_1(2u+iK')/L\eta}h(u) \\ h(u+2K) &= -e^{2i\pi m_2(u+K)/L\eta}h(u) \end{aligned} \tag{120}$$

This work was re-derived by Takhtadzhan and Faddeev[39] using the "Quantum Inverse Scattering Method" (QISM). Then in 1982 the author presented the functional relation method in Sections (10.5) and (10.6) of his book.[27] An explicit construction (for N even) of the matrix $\tilde{Q}(v)$ for the eight-vertex model is given in Section 6 of ref. 31, and in Section (10.5) of ref. 27.

The notation in ref. 27 is slightly different from that in I, II, III and ref. 39. The restriction (113) is not made, $H(u), \Theta(u)$ are the standard theta functions, the elliptic integrals K, K' are written as I, I', and if we write λ, v therein as λ_B, v_B, then[13]

$$\lambda_B = 2i(K-\eta), \qquad v_B = 2i(v-K) \tag{121}$$

As always, there are errors and inconsistencies. This is a good opportunity to correct two of them.

One is a simple but significant typographical error: $j+1$ in Eq. (10.5.8) should be $j-1$, so that it should read

$$s_j = s + \lambda(\sigma_1 + \cdots + \sigma_{j-1}) \tag{122}$$

Equation (10.5.21) is then consistent with (I.78).[14]

The other is an omission (or at least an over-simplification) by the author in the Bethe ansatz derivation of eigenvectors in papers I–III. In (7) of ref. 28 each eigenvalue $Q(v)$ of the matrix $\tilde{Q}(v)$ is taken to be simply a product of elliptic theta functions. This is corrected in Eq. (6.10) of ref. 29, and in (10.6.8) of ref. 27, where an exponential factor is also included, so that

$$Q_B(v) = e^{2i\tau v} \prod_{j=1}^{N/2} H_{Jb}(v-u_j)\, \Theta_{Jb}(v-u_j) \tag{123}$$

where the suffix B is inserted to distinguish this function from that of III, τ and $u_1, \ldots, u_{N/2}$ satisfy

$$\tau = \pi(\hat{s} - 1 + N + 4p')/8K \tag{124}$$

$$u_1 + \cdots + u_{N/2} = (\hat{r}\hat{s} - 1 + 4p)\, K/2 - i(\hat{s} - 1 + N + 4p')\, K'/4 \tag{125}$$

Here p, p' are integers, and $\hat{r} = \pm 1$ is the eigenvalue of the operator R that reverses all arrows (or spins), and $\hat{s} = \pm 1$ depending on whether the number of down arrows is even or odd (and S is the diagonal matrix with entries $\hat{s}$). We have used the notation of (10.6.7)–(10.6.8) of ref. 27, except that we have converted from the $v = v_B$ therein to the present v, u_j (which are those of papers I–III) by (121).

[13] This still leaves d with a different sign in ref. 27 from that in I, II, III, but since vertices 7 and 8 are sinks and sources of arrows, changing the sign of d does not affect the partition function or the eigenvalues of the transfer matrix.

[14] Apart from $\pm i$ factors that presumably arise because of the negation of d.

For $\tilde{n} = N/2$, the eigenvalue equation (III.1.21) should be the same as (10.6.1) of ref. 27. After converting to the notation of III, we find that it is the same, and is consistent with the other equations (III.1.1)–(III.1.23), if extra ω factors are included in (III.1.14), (III.1.21), (III.1.23) to make them become

$$\Psi = \sum_{l=1}^{L} \sum_{\mathbf{X}} \omega^l f(l \mid x_1,..., x_{\tilde{n}})\, \psi(l_1,..., l_{N+1}) \tag{126}$$

$$\Lambda = \omega\phi(v-\eta) \prod_{j=1}^{\tilde{n}} \frac{h(v-u_j+2\eta)}{h(v-u_j)} + \omega^{-1}\phi(v+\eta) \prod_{j=1}^{\tilde{n}} \frac{h(v-u_j-2\eta)}{h(v-u_j)} \tag{127}$$

$$\omega^{-2} e^{iNk_j} = -\prod_{m=1}^{\tilde{n}} h(u_j-u_m+2\eta)/h(u_j-u_m-2\eta) \tag{128}$$

Here

$$\omega = e^{2\pi i\tilde{m}/L} \tag{129}$$

where the integer $\tilde{m}$ is given by

$$2\tilde{m} = m_1(\hat{s}-1+N+4p') + m_2(\hat{r}\hat{s}-1+4p) \tag{130}$$

The other equations amongst (III.1.1)–(III.1.22) remain unaffected, in particular e^{ik_j} is defined by (III.1.17):

$$e^{ik_j} = h(u_j+\eta)/h(u_j-\eta) \tag{131}$$

We can still define a function $Q(v)$ by (III.1.24):

$$Q(v) = \prod_{j=1}^{\tilde{n}} h(v-u_j) \tag{132}$$

but it differs from $Q_B(v)$ via the exponential factors in (115) and (123). Then, exhibiting the dependence of the transfer matrix eigenvalue Λ on v, (127) can be written as

$$\Lambda(v)\, Q(v) = \omega\phi(v-\eta)\, Q(v+2\eta) + \omega^{-1}\phi(v+\eta)\, Q(v-2\eta) \tag{133}$$

which equation replaces (III.1.25).

These are in fact the correct equations, not just for $\tilde{n}=N/2$, but for all $\tilde{n}$ satisfying (114).[15] We refer to Eqs. (III.1.1)–(III.1.25), with the replacements (126)–(133), as the *corrected eight vertex Bethe ansatz equations.*

There was an omission in the derivation in paper III. Equation (III.2.2) is correct as written, but in (III.3.1) the author should have allowed the more general ansatz of including a factor ω^l in the rhs, where $\omega^L=1$. This is equivalent to associating this factor with $f(l \mid x_1,..., x_{\tilde{n}})$ and multiplying the first term on the rhs of (III.2.2) by ω, the second by ω^{-1}, and (via the "wanted terms" and the "unwanted boundary terms" of Section 3 of III) to the introduction of the ω factors in (126)–(133).

Precisely these ω factors are included in the $\tilde{n}=0$ equations (II.5.6) and (II.5.7), where $g_1=\phi(v+\eta)$ and $g_2=\phi(v-\eta)$. They are also included in the work of Takhtadzhan and Faddeev.[(39)] [16]

As Takhtadzhan and Faddeev comment [39, after (5.26)], the original equations of paper III appear to apply only for the case when $\tilde{m}=0$. However, it is better than that. One can verify, for all values of $\tilde{n}$ satisfying (114), using (120), that the corrected eight vertex Bethe ansatz equations and (130) are unaffected (apart from normalization factors that merely renormalize the eigenvector Ψ), by the following simultaneous substitutions:

$$u_1 \to u_1+iK', \qquad \omega \to e^{-4i\pi m_1/L}\omega$$

$$\tilde{m} \to \tilde{m}-2m_1, \qquad p' \to p'-1$$

Similarly, if u_1 is incremented by $2K$, then $\tilde{m}$, p are incremented by $2m_2$, 1, respectively.

The same remarks apply if u_1 is replaced by any of the $u_1,..., u_{\tilde{n}}$.

We can use this freedom, which is simply the choice of period parallelograms for the zeros $u_1,..., u_{\tilde{n}}$, to increment $\tilde{m}$ by any even integer. If L is odd, this means that we can construct any choice of ω by such shifts; while if L is even, we can construct half the choices. So the 1973 papers do cover approximately three-quarters of the cases!

More recently, related equations for the eigenvectors of the eight-vertex model have been studied by Felder and Varchenko,[(41)] and by Deguchi.[(42)]

[15] Except that the restrictions (125), (130) apply only for $\tilde{n}=N/2$, which can be regarded as the generic case, and not necessarily even then if complete strings are present: we return to this point later in the section.

[16] Our ω is $e^{-2\pi im/Q}$ in the notation of Takhtadzhan and Faddeev, and it seems that the l of our equation (126) must be the $-l$ of their equation (5.23).

Alternative Form of Bethe's Equations

From (120) and (132), remembering that N is even,

$$\begin{aligned} Q(v+\hat{u}+iK') &= (-1)^{\tilde{n}}\, e^{-i\pi\tilde{n}m_1(2v+iK')/L\eta} Q(v+\hat{u}) \\ Q(v+\hat{u}+2K) &= (-1)^{\tilde{n}}\, e^{2i\pi\tilde{n}m_2(v+K)/L\eta} Q(v+\hat{u}) \\ \phi(v+iK') &= e^{-i\pi N m_1(2v+iK')/L\eta}\phi(v) \\ \phi(v+2K) &= e^{2i\pi N m_2(v+K)/L\eta}\phi(v) \end{aligned} \tag{134}$$

where

$$\hat{u} = (u_1 + \cdots + u_{\tilde{n}})/\tilde{n} \tag{135}$$

From (116), or from (114) and (133), the function $\Lambda(v)$ satisfies the same quasi-periodicity relations as those above for $\phi(v)$.

The functions $Q(v), \phi(v), \Lambda(v)$ are all entire. Using the $v \to v+2K$ quasi-periodicities above, it follows that there exist coefficients q_j, f_j, t_j such that

$$\begin{aligned} Q(v) &= e^{i\pi\tilde{n}m_2(v-\hat{u})^2/2KL\eta} \sum_j (-1)^j\, q_j e^{i\pi j'(v-\hat{u})/K} \\ \phi(v) &= e^{i\pi N m_2 v^2/2KL\eta} \sum_j f_j e^{i\pi j v/K} \\ \Lambda(v) &= e^{i\pi N m_2 v^2/2KL\eta} \sum_j t_j e^{i\pi j v/K} \end{aligned} \tag{136}$$

where j takes all positive and negative integer values and $j' = j$ if $\tilde{n}$ is even, while $j' = j - 1/2$ if $\tilde{n}$ is odd.

Now using the $v \to v+iK'$ quasi-periodicities, we find that the coefficients in these series must satisfy

$$\begin{gathered} q_{j+\tilde{n}} = e^{-\pi(2j'+\tilde{n})\,K'/2K} q_j \\ f_{j+N} = e^{-\pi(2j+N)\,K'/2K} f_j, \qquad t_{j+N} = e^{-\pi(2j+N)\,K'/2K} t_j \end{gathered} \tag{137}$$

If $q_0, \ldots, q_{\tilde{n}-1}$ are known, then the simple periodicity relation (137) determines all the other q_j. Similarly, all the t_j are determined by $t_0, \ldots, t_{N-1}$, and the known f_j by $f_0, \ldots, f_{N-1}$.

These series are convergent for all finite v. Substituting them into (133) and equating coefficients, we obtain

$$\sum_{m=-\infty}^{\infty} B_{j,m} q_m = 0 \tag{138}$$

for all integers j, where

$$B_{j,m} = -t_{j-m} + e^{i\pi m_2(8\tilde{n}-N)\eta/2LK}[\tilde{\omega} f_{j-m+m_2v'} + \tilde{\omega}^{-1} f_{j-m-m_2v'}]$$

where

$$\tilde{\omega} = \omega e^{i\pi(m+2m'-j)\eta/K} e^{-2i\pi\tilde{n}m_2\hat{u}/LK}$$

and $m' = m$ if $\tilde{n}$ is even, $m' = m - 1/2$ if $\tilde{n}$ is odd.

Define

$$\bar{q}_j = e^{\pi j'^2 K'/2\tilde{n}K} q_j \tag{139}$$

$$\bar{B}_{j,m} = (-1)^{j+m} e^{\pi j'^2 K'/2(N+\tilde{n})K} e^{-\pi m'^2 K'/2\tilde{n}K} B_{j,m} \tag{140}$$

Then (138) becomes

$$\sum_{m=-\infty}^{\infty} (-1)^m \bar{B}_{j,m} \bar{q}_m = 0 \tag{141}$$

and $\bar{q}_m$, $\bar{B}_{j,m}$ satisfy the periodicity relations

$$\bar{q}_{m+\tilde{n}} = \bar{q}_m, \qquad \bar{B}_{j+N+\tilde{n},m+\tilde{n}} = \bar{B}_{j,m} \quad \forall m, j \tag{142}$$

We can therefore define discrete Fourier transforms $\hat{q}_\alpha$, $\hat{B}_{j,\alpha}$ such that

$$\bar{q}_m = \sum_{\alpha=0}^{\tilde{n}-1} e^{2i\pi\alpha m/\tilde{n}} \hat{q}_\alpha, \qquad \hat{B}_{j,\alpha} = \sum_{m=-\infty}^{\infty} e^{2i\pi\alpha m/\tilde{n}} \bar{B}_{j,m} \tag{143}$$

m, α being integers in the ranges $-\infty < m < \infty$ and $0 \leqslant \alpha < \tilde{n}$. Then (141) becomes

$$\sum_{\alpha=0}^{\tilde{n}-1} \hat{B}_{j,\alpha} \hat{q}_\alpha = 0 \tag{144}$$

Here j can take any integer value, but from (142)

$$\hat{B}_{j+N+\tilde{n},\alpha} = \hat{B}_{j,\alpha} \tag{145}$$

so there is no loss of information in restricting j in (144) to lie in the range

$$0 \leqslant j < N + \tilde{n}$$

Hence (144) can be regarded as a set of $N+\tilde{n}$ homogeneous linear equations for the $\tilde{n}$ unknowns $\hat{q}_\alpha$. It is also linear in the coefficients t_j, and all of these can be determined immediately and linearly from (137) if we know $t_0,\ldots, t_{N-1}$. Thus $t_0,\ldots, t_{N-1}$ play the role of generalized eigenvalues, as $t_0,\ldots, t_N$ do for the six-vertex model in (41), (42). Since only the ratios of the $\hat{q}_\alpha$ enter the equation, we have $N+\tilde{n}$ equations for a total of $N+\tilde{n}-1$ unknowns. Unlike the six-vertex model, this set is over-determined, presumably because of the quasi-periodicity constraints satisified by the elliptic functions. They must of course have solutions.

Many of the remarks made following equation (38) for the six-vertex model extend to the eight-vertex model. One can construct an $N+\tilde{n}$ by $\tilde{n}$ matrix $\mathbf{B}$ with elements $\hat{B}_{j,\alpha}$, and write (144) as $\mathbf{B}\hat{\mathbf{q}}=\mathbf{0}$. Thus $\mathbf{B}$ must have rank at most $\tilde{n}-1$, and $\hat{\mathbf{q}}$ is the column null vector of $\mathbf{B}$.[17] If the eigenvalue $\Lambda(v)$ is degenerate (for all v), then the rank of $\mathbf{B}$ will be less than $\tilde{n}-1$. There will then be more than one solution for $\hat{\mathbf{q}}$, and hence of (133) for $Q(v)$. One can expect such behaviour in the situation that we shall now discuss, i.e., when $Q(v)$ contains one or more complete strings.

Similarities to the Six-Vertex Model: Strings

The eight-vertex model is a generalization of the zero-field six-vertex model, and many of the remarks we have made about the six-vertex model continue to apply. There are still very special eigenvectors with eigenvalues of the form (57), namely the eigenvectors discussed in papers I and II, corresponding to $\tilde{n}=0$ in (127). For N even, explicit expressions for the matrix $\tilde{Q}$ are given in ref. 29 and in Section 10.5 of ref. 27. From these it follows that each eigenvalue $Q(v)$ is a product of $N/2$ elliptic $h(u)$ functions, so the only way we can get these simple eigenvalues is for all the $u_1,\ldots, u_{N/2}$ to be grouped into complete strings. These are of length M, where

$$\begin{aligned} M &= L/2 \qquad \text{if } L \text{ is even} \\ &= L \qquad \text{if } L \text{ is odd} \end{aligned} \tag{146}$$

Each string consists of M zeros, say $u_1,\ldots, u_M$, differing sequentially by 2η, i.e., for $j=1,\ldots, M$

$$u_{j+1}=u_j+2\eta \tag{147}$$

interpreting u_{M+1} as u_1, modulo $2M\eta$.

In fact any eigenvector and eigenvalue given by (III.1.1)–(III.1.23)[18] with $\tilde{n}\neq N/2$, must also be present in the case $\tilde{n}=N/2$, differing from it

[17] It is a compication that $\mathbf{B}$ depends on $\hat{u}$. This disappears if $m_2=0$: possibly it can be removed for other values by using the periodicity in integer multiples of η (rather than $2K$) as the basis for the discrete Fourier transforms.

[18] Including the modifications (126)–(128).

only in the subtraction or addition of complete strings. The eigenvectors for $\tilde{n} \neq N/2$ lie in the eigenspace of any other allowed value of $\tilde{n}$ which is closer to (or equal to) $N/2$, at least for $\tilde{n} \leqslant N/2$.

For *any* values of L, m_1, m_2, the condition (114) is satisfied by $\tilde{n} = N/2$, so this is the generic case. The eigenvalue equations (127)–(131) then apply for all η. The eigenvector equations, notably (126), depend on η satisfying (113), but one can approach arbitrarily close to any desired value by taking L, m_1, m_2 sufficiently large.[19]

Other values of $\tilde{n}$ only occur if (113) is satisfied for L not greater than N.

Again, there are technical difficulties about handling Bethe's equations when there are complete strings. The remarks of Section 4 extend from the six-vertex to the eight-vertex model. One should replace (128) by the two equations

$$\omega^{-2} e^{iNk_j} = \prod_{m=1, m \neq j}^{\tilde{n}} (-t_{m,j}/t_{j,m}) \tag{148}$$

$$t_{j,m} h(u_j - u_m + 2\eta) = t_{m,j} h(u_m - u_j + 2\eta) \tag{149}$$

and write (III.1.20) as

$$A(P) = \epsilon_P C^{-1} \prod_{1 \leqslant j < m \leqslant \tilde{n}} t_{Pm, Pj} \tag{150}$$

where the renormalization factor C is the same for all permutations P and is the product of selected $t_{j,m}$ factors, one from each string, for which $h(u_m - u_j + 2\eta)$ vanishes.

If u_j does not belong to any string, then in (148) we can take $t_{j,m} = h(u_m - u_j + 2\eta)$. The contribution to the rhs from the u_m that do lie within strings cancels out, leaving a reduced equation where j, m only take non-string values. If u_j does belong to a string, say to (147), then $t_{12}, t_{23}, \ldots, t_{M,1}$ vanish but their ratios remain finite. If one fixes one of the u_j within the string, then the rest are determined, the e^{ik_j} are given by (131), and the ratios of $t_{12}, t_{23}, \ldots, t_{M,1}$ are determined by (148) for $j = 1, \ldots, M$. From (150), some of the coefficients $A(P)$ involve $t_{12}, t_{23}, \ldots, t_{M,1}$ only via these ratios, so are finite and non-zero. The other $A(P)$ are of linear or higher order in $t_{12}, t_{23}, \ldots, t_{M,1}$, so vanish.

Again, one is free to vary each of the string centres at will. As one varies these parameters, and the disposable parameters s, t in III, the eigenvector Ψ will trace out a surface $\mathscr{S}$ in the eigenspace of the eigenvalue Λ. If the eigenvalue is unique, these variations will merely change the normalization.

[19] The sum over l in (126) is a discrete Fourier transform: for general values of η it may be appropriate to replace it by a continuous one.

If it is degenerate, $\mathscr{S}$ will lie in the eigenspace appropriate to this value of $\tilde{n}$, and we expect the vectors on $\mathscr{S}$ to span this eigenspace.

Since the explicit construction of $\tilde{Q}(v)$ given in Section 10.6 of ref. 27 gives $\tilde{n} = N/2$, we expect this to be the generic case, giving all eigenvalues and a complete set of eigenvectors of $T(v)$. For $\tilde{n}$ satisfying (114), but not equal to $N/2$, one expects to only observe the degenerate eigenvalues, and to obtain only a sub-space of the eigenspace of each. Recent numerical results support these expectations.[43]

Of course, one may have particular reasons for fixing the string centres at particular values, as Fabricius and McCoy did for the six-vertex model.[14–17] An obvious choice (for $\tilde{n} = N/2$) is to fix them so that $Q(v)$ is the eigenvalue of the matrix $\tilde{Q}(v)$ constructed in Section 6 of ref. 29 and in Section (10.5) of ref. 27. Equivalently, one can require that they be fixed to their limiting values (again, for $\tilde{n} = N/2$) as η approaches the "root of unity" value (113). However, these considerations lie outside the Bethe ansatz for a fixed value of η. The Bethe ansatz is complete without them: the arbitrariness in the string centres (and in s, t) is a reflection of the degeneracy of the eigenvalues of the tranfer matrix, and the resulting non-uniqueness of the eigenvectors.

The Six-Vertex Model Limit

In the limit when the elliptic modulus k (or the nome q) goes to 0, 1 or ∞, the elliptic functions become trigonometric functions and the eight-vertex model becomes the zero-field six-vertex model. Much of the working of this section can be adapted at once to the six-vertex model, except that some of the $u_1, \ldots, u_{N/2}$ may become infinite. In this way the resulting six-vertex model function $Q(v)$ can have any number $n \leqslant N/2$ of finite zeros, and the ω factors in (127) can be related to those in (73). It should be possible to obtain all the six-model eigenvalues from the those of the eight-vertex model by taking such a limit.

There is a problem with the eigenvectors. For $n \neq N/2$, the zero-field six-vertex model eigenvalues occur in degenerate pairs, one in the sub-space with n down arrows, the other in the arrow-inverted sub-space with $N-n$ down arrows. Thus two eight-vertex model eigenvalues, with opposite spin-reversal symmetry, must coalesce. Their sum and difference will then be the six-vertex model eigenvectors in the two sub-spaces. Only in the $n = N/2$ sub-space can one expect to obtain the six-vertex model eigenvectors directly as limits of those of the eight-vertex model.

The Sum Rule

The constraint (125) applies to the eight-vertex $Q(v)$ functions obtained by the explicit construction in Section (10.6) of ref. 27, which have $\tilde{n}=N/2$ zeros. If the eigenvalue $\Lambda(v)$ is non-degenerate, i.e., if there are no exact complete strings, then $Q(v)$ is uniquely defined by Beth's equations, so $\tilde{n}$ will be equal to $N/2$, and (125) will automatically be satisfied.

However, if there are strings, even if $\tilde{n}=N/2$, then one can shift the string centres arbitrarily and Bethe's equations will be unaffected. The resulting eigenvector will not necessarily be an eigenvector of R and S, and (125) will not in general be satisfied, but it will nevertheless be a valid eigenvector of the transfer matrix $T(v)$.

So we conclude that, for all η satisfying (113), a complete set of 2^N eigenvalues and eigenvectors can be obtained by taking $\tilde{n}=N/2$ and observing the constraint (125) (though even then the eigenvectors will not necessarily also be eigenvectors of R and S for all values of the string centres and the disposable parameters s and t[43]). Further eigenvectors can be obtained by abandoning these constraints, while of course retaining (114), but these eigenvectors will lie in the eigenspaces obtained with the constraints, so do not extend these eigenspaces.

These remarks extend to the six-vertex model limit. In the $n=N/2$ subspace, non-degenerate eigenvalues must satisfy the analogue of (125), which, using (121), is

$$v_1+\cdots+v_{N/2}=i\pi(rs-1+N+4p)/2 \tag{151}$$

These $v_1,\ldots, v_{N/2}$ are those of Sections 1 through 6, p is an arbitrary integer, r the eigenvalue (± 1) of the spin reversal operator, and $s=(-1)^{N/2}$ the arrow parity of this state, with $N/2$ down arrows. This is Eq. (17) of ref. 44, and Eq. (1.44) of ref. 17. Again, if the eigenvalue is degenerate and $v_1,\ldots, v_{N/2}$ contain one or more complete strings, then (151) will not necessarily be satisfied. I am indebted to Barry McCoy and Klaus Fabricius for correspondence on this and related matters concerning the Bethe ansatz.

Other Possible Difficulties: Bound Pairs

Because of the double periodicities of elliptic functions, the technical problems in the six-vertex model associated with zeros $v_1,\ldots, v_n$ going to infinity cannot occur in the eight-vertex model: the corresponding $u_1,\ldots, u_{\tilde{n}}$ can be restricted to a period parallelogram. There appears to be no "beyond the equator" problem. All the states are accounted for by taking $\tilde{n}=N/2$. The equations may well have solutions for $\tilde{n}>N/2$: this would correspond to adding complete strings to $Q(v)$.

One could still have bound pairs analogous to (76), when $-u_1=u_2=\eta$,

$$e^{ik_1}=e^{-ik_2}=0, \qquad e^{i(k_1+k_2)}=-1 \tag{152}$$

and the ratio t_{21}/t_{12} vanishes. One would expect to handle this in the same way as for the six-vertex model, using (148) for $j=1,2$ to calculate $e^{iNk_1}t_{12}/t_{21}$ and $e^{iNk_2}t_{21}/t_{12}$, then substituting these into the suitably renormalized equations for $A(P)$ and the eigenvector Ψ.

Although we have not observed the phenomenon, it is conceivable that two or more of $u_1,\ldots, u_{\tilde{n}}$, say $u_1,\ldots, u_p$, could coincide at some arbitrary value. This would require either generalizing the argument of Section 6, or first dividing (III.1.16) by the product of $h(u_i-u_j)$ over $1\leqslant i<j\leqslant p$, and then taking the limit where $u_1,\ldots, u_p$ become equal.

8. SUMMARY

We have presented the coordinate Bethe ansatz equations with some care, trying to avoid (or at least signpost) the problems that occur when some of the variables are zero or infinity. Perhaps the essential point of this paper is that for the six-vertex model the Bethe ansatz equations are (5)–(17). It seems that one can always choose the coefficients $A(P)$ to be given by (43), but this is not necessary if enough of the s_{ij} vanish (as they do when the v_j are equal and infinite). All that is necessary for g to be an eigenvector is that (5)–(17) (or their appropriately renormalized forms) be satisfied, with $g\neq 0$.

For the six-vertex model, in Section 3 we have discussed the situation that arises when some of the Bethe zeros v_j are infinite, and how this leads to a reduced Bethe equation containing ω factors. In Section 4 we show that this is the key to resolving the "beyond the equator" problem and to constructing the Bethe eigenvector for $n>N/2$. We also show how to cope with the problem of a bound pair, when two of the momenta are infinite but their sum is finite (equal to an odd integer multiple of $i\pi$).

In Section 5 we look at the problem discussed by Fabricius and McCoy, when there are one or more exact complete strings. We show that the v_j are no longer uniquely determined, because the string centres can be chosen at will. Nevertheless, the Bethe ansatz equations are satisfied, and we show how to construct the (necessarily non-zero) eigenvector by working with appropriate ratios of the vanishing t_{ij}.

In particular, we have found the solutions of the Bethe ansatz corresponding to all the $v_1,\ldots, v_n$ lying on a single complete string. The Bethe ansatz equations do not define the string centre (the average of $v_1,\ldots, v_n$). This is to be expected: it is a direct consequence of the eigenvalue Λ being degenerate, which means that there is more than one eigenvector, and hence more than one solution of the Bethe ansatz. We show that the ansatz can be used to construct a *complete* set of eigenvectors, spanning the eigenspace.

One should of course go on to study more complicated situations, where there are more than one complete strings, and not all the v_j belong to a string. We expect the above methods to generalize to such cases, but the algebra may well be complicated.

In all the cases dealt with in Sections 3 to 5, one can always use (5) (with an appropriate choice of the normalization factor C in (43)) as written, each term in the summand being finite and the sum being finite and non-zero. The only "limit" is that of recognizing that one has the set of rational equations (20), (44)–(46) to solve for the variables e^{v_j}, e^{ik_j} and t_{ij}. Some of these variables may be zero or infinite, while what one wants is their ratios or some other product of powers, which are finite and non-zero. These ratios or products can of course themselves be regarded as variables. This is only a generalization of the usual practice of including the "point at infinity" in the domain of the variables.

In Section 6 we touch on another problem, namely what happens if two or more of the v_j are equal and finite. (The question of what happens when they are equal and infinite is different, actually easier to resolve, and is dealt with in Section 3.) In general this case really does seem to demand that one take a limit in the expression (5) for the eigenvector g, since the terms in the summand are finite and non-zero but cancel one another in pairs, so that their sum is zero. This is different from and less satisfactory than the other cases, but we remark that in fact we never encountered this case in our numerical experiments, and it is not the case discussed in refs. 9–17. It is not clear that it ever actually occurs.

In Section 7 we discuss the zero-field eight-vertex model with an even number of columns, and give the needed corrections to the coordinate Bethe ansatz equations of Section 1 of ref. 38. We indicate how the string and infinite momenta problems can occur also for this model, and how to resolve them.

If one wants to make a specific choice of the string centres, particularly if one wants to ensure continuity as λ or η passes through the "root of unity" value (80) or (113), or (equivalently) if one wants $Q(v)$ to be the eigenvalue of the matrix $\tilde{Q}(v)$ of (86) of ref. 31, or (10.5.31) of ref. 27, then clearly one should use the results of Fabricius and McCoy. They have addressed this problem in a series of well-presented papers, and have systematically exhibited the connections to the sl_2 loop algebras. However, they do use the provocative title "Bethe's equation is incomplete...."[15] If all one wants to do is to diagonalize the transfer matrix (or the XXZ hamiltonian), obtaining all the eigenvalues, their degeneracies and eigenspaces, then it seems that there is no need to look further than the Bethe ansatz. At least for the cases studied in this paper, the Bethe ansatz is in fact complete.

The alternative forms (41), (138) of Bethe's equations are themselves generalized eigenvalue equations in which the $n+1$ or $\tilde{n}$ independent coefficents q_j are the elements of the eigenvector. They have some advantages over (44), (45), (148), (149), being linear in the q_j and the t_j. The "infinite v_j" problem merely corresponds to some of the coefficients q_j vanishing.[20] It seems surprising, but while the author did not believe these equations to be new, he has been unable to find any previous paper where they have been written down.

APPENDIX A

Suppose for the moment that $A(1,..., n)$ is non-zero. Let $\alpha_{m,j}$ be $A(P_{m,j})$, where $P_{m,j}$ is the permutation where j is removed from its place in the sequence $1, 2,..., n$ and replaced immediately after m, or immediately before $m+1$. Thus

$$\begin{aligned} \alpha_{m,j} &= A(1,..., m, j, m+1,..., j-1, j+1,..., n) \qquad \text{if} \quad m < j \\ \alpha_{m,j} &= A(1,..., j-1, j+1,..., m, j, m+1,..., n) \qquad \text{if} \quad m \geqslant j \end{aligned} \tag{A.1}$$

In particular,

$$\alpha_{0,j} = A(j, 1,..., j-1, j+1,..., n), \qquad \alpha_{j-1,j} = \alpha_{j,j} = A(1,..., n)$$
$$\alpha_{n,j} = A(1,..., j-1, j+1,..., n, j)$$

Fix j at some value between 1 and n. Then there are $n-1$ equations of the set (7) that involve only the n distinct coefficients $\alpha_{0,j},..., \alpha_{n,j}$, namely

$$s_{m,j}\alpha_{m,j} + s_{j,m}\alpha_{m-1,j} \tag{A.2}$$

for $m = 1,..., n$, $m \neq j$. Also, from (8) we find that

$$e^{iNk_j}\alpha_{n,j} = \alpha_{0,j} \tag{A.3}$$

Together, these give us n linear homogeneous equations in n unknowns. Since at least one of the unknowns, namely $\alpha_{j-1,j} = \alpha_{j,j} = A(1,..., n)$, is non-zero, the determinant of the matrix of coefficients of these n equations must vanish. This determinant is easily obtained, giving

$$e^{iNk_j} \prod_{m=1, m\neq j}^{n} s_{j,m} = (-1)^{n-1} \prod_{m=1, m\neq j}^{n} s_{m,j} \tag{A.4}$$

and this must hold for all the possible values $1,..., n$ of j.

[20] But in general one still needs to calculate the zeros v_j or u_j in order to obtain the eigenvector.

Permuting the indices 1,..., n merely rearranges the n equations (A.4), so our initial assumption that $A(1,..., n)$ is non-zero is irrelevant: to derive (A.4) it is sufficient that any one of the coefficients $A(p_1,..., p_n)$ be non-zero. This must be so for g to be a non-zero vector.

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REFERENCES

1. H. Bethe, Zur Theorie der Metalle, *Z. Physik* **71**:205–226 (1931); translated in *The Many-Body Problem*, D. C. Mattis, ed. (World Scientific, Singapore, 1993), pp. 689–716.
2. A. N. Kirillov, Combinatorial identities, and completeness of eigenstates for the Heisenberg magnet, *J. Soviet Mathematics* **30**:2298–2310 (1985); translated from *Zap. Nauch. Sem. LOMI* **131**:88–105 (1983).
3. A. N. Kirillov and N. A. Liskova, Completeness of Bethe's states for the generalized XXZ model, *J. Phys. A* **30**:1209–1226 (1997).
4. A. Klümper and J. Zittartz, The eight-vertex model: Spectrum of the transfer matrix and classification of the excited states, *Z. Phys. B Condensed Matter* **75**:371–384 (1989).
5. F. H. L. Essler, V. E. Korepin, and K. Schoutens, Complete solution of the one-dimensional Hubbard model, *Phys. Rev. Lett.* **67**:3848–3851 (1991).
6. F. H. L. Essler, V. E. Korepin, and K. Schoutens, Fine structure of the Bethe ansatz for the spin-1/2 Heisenberg XXX model, *J. Phys. A* **25**:4115–4126 (1992).
7. G. Juettner and M. Karowski, Completeness of "good" Bethe ansatz solutions of a quantum group invariant Heisenberg model, *Nucl. Phys. B* **430**:615–632 (1994).
8. A. Kuniba and T. Nakanishi, The Bethe equation at $q=0$, the Möbius inversion formula and weight multiplicities I: The $sl(2)$ case, Progress in Mathematics, in *Physical Combinatorics*, Vol. 191, M. Kashiwara and T. Miwa, eds. (Birkhauser, Boston, 2000), pp. 185–216.
9. L. D. Faddeev and L. A. Takhtadzhyan, Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model, *Zap. Nauch. Sem. LOMI* **109**:134–178 (1981); translated in *J. Sov. Math.* **24**:241–267 (1984).
10. G. P. Pronko and Yu. G. Stroganov, Bethe equations "on the wrong side of the equator," *J. Phys. A Math. Gen.* **32**:2333–2340 (1999).
11. R. Siddharthan, Singularities in the Bethe solution of the XXX and XXZ Heisenberg spin chains, cond-mat/9804210.
12. A. Wal, T. Lulek, B. Lulek, and E. Kozak, The Heisenberg magnetic ring with 6 nodes: Exact diagonalization, Bethe ansatz and string configurations, *Int. J. Mod. Phys. B* **13**:3307–3321 (1999).
13. J. D. Noh, D.-S. Lee, and D. Kim, Origin of the Singular Bethe ansatz solutions for the Heisenberg XXZ spin chain, cond-mat/0001175.
14. T. Deguchi, K. Fabricius, and B. M. McCoy, The sl_2 loop algebra symmetry of the six-vertex model at roots of unity, *J. Statist. Phys.* **102**:701–736 (2001).

15. K. Fabricius and B. M. McCoy, Bethe's equation is incomplete for the XXZ model at roots of unity, *J. Statist. Phys.* **103**:647–678 (2001).
16. K. Fabricius and B. M. McCoy, Completing Bethe's equations at roots of unity, *J. Statist. Phys.* **104**:573–587 (2001).
17. K. Fabricius and B. M. McCoy, Evaluation parameters and Bethe roots for the six-vertex model at roots of unity, LANL pre-print condmat/010857.
18. T. Deguchi, The 8V CSOS model and the sl_2 loop algebra symmetry of the six-vertex model at roots of unity, LANL pre-print condmat/0110121.
19. E. H. Lieb, Exact solution of the problem of the entropy of two–dimensional ice, *Phys. Rev. Lett.* **18**:692–694 (1967).
20. E. H. Lieb, Exact solution of the F model of an antiferroelectric, *Phys. Rev. Lett.* **18**:1046–1048 (1967).
21. E. H. Lieb, Exact solution of the two-dimensional Slater KDP model of a ferroelectric, *Phys. Rev. Lett.* **19**:108–110 (1967).
22. E. H. Lieb, Residual entropy of square ice, *Phys. Rev.* **162**:162–172 (1967).
23. B. Sutherland, Exact solution of a two-dimensional model for hydrogen bonded crystals, *Phys. Rev. Lett.* **19**:103–104 (1967).
24. C. P. Yang, Exact solution of two dimensional ferroelectrics in an arbitrary external field, *Phys. Rev. Lett.* **19**:586–588 (1967).
25. B. Sutherland, C. N. Yang, and C. P. Yang, Exact solution of two dimensional ferroelectrics in an arbitrary external field, *Phys. Rev. Lett.* **19**:588–591 (1967).
26. R. J. Baxter, Generalized ferroelectric model on a square lattice, *Studies in Applied Mathematics (M.I.T.)* **50**:51–69 (1971).
27. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).
28. R. J. Baxter, Eight-vertex model in lattice statistics, *Phys. Rev. Lett.* **26**:832–833 (1971).
29. R. J. Baxter, Partition function of the eight-vertex lattice model, *Ann. Phys. (N.Y.)* **70**:193–228 (1972).
30. R. J. Baxter, S. B. Kelland, and F. Y. Wu, Equivalence of the Potts model or Whitney polynomial with an ice-type model, *J. Phys. A Math. Gen.* **9**:397–406 (1976).
31. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors, *Ann. Phys. (N.Y.)* **76**:1–24 (1973).
32. V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, Integrable structure of conformal field theory: II. Q-operator and DDV equations, *Comm. Math. Phys.* **190**:247–278 (1997).
33. V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, Integrable structure of conformal field theory: III. The Yang–Baxter relation, *Comm. Math. Phys.* **200**:297–324 (1999).
34. M. T. Batchelor, *Finite Lattice Methods in Statistical Mechanics*, Ph.D. thesis (Australian National University, Canberra, 1987).
35. On the spectrum of the XXZ-chain at roots of unity, *J. Statist. Phys.* **105**:607–709 (2001).
36. R. B. Jones, Baxter's method for the XXZ model, *J. Phys. A* **7**:495–504 (1974).
37. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. II. Equivalence to a generalized ice-type lattice model, *Ann. Phys. (N.Y.)* **76**:25–47 (1973).
38. R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. III. Eigenvectors of the transfer matrix and Hamiltonian, *Ann. Phys. (N.Y.)* **76**:48–71 (1973).
39. L. A. Takhtadzhan and L. D. Faddeev, The quantum method of the inverse problem and the Heisenberg XYZ model, *Uspekhi Mat. Nauk* **34(5)**:13–63 (1979); translated in *Russian Math. Surveys* **34(5)**:11–68 (1979).
40. K. Fabricius and B. M. McCoy, private communication.

41. G. Felder and A. Varchenko, Algebraic Bethe ansatz for the elliptic quantum group $E_{\tau,\eta}(sl_2)$, *Nucl. Phys. B* **480**:485–503 (1996).
42. Construction of some missing eigenvectors of the XYZ spin chain at the discrete coupling constants and the exponentially large spectral degeneracy of the transfer matrix, LANL pre-print condmat/0109078.
43. K. Fabricius, private communication.
44. A. Doikou and R. I. Nepomechie, Discrete symmetries and *S* matrix of the XXZ chain, *J. Phys. A* **31**:L621–L627 (1998).